GEOMETRY

Optimization via Geometric Transformations

(1) **Warm-up:** Consider two circles Γ_1 and Γ_2 which intersect at points A and B. Let MN be a line passing through A, with the point M on Γ_1 and the point N on Γ_2. Consider two parallel lines MC and ND, where MC is a chord for the circle Γ_1 and ND is a chord for the circle Γ_2. What can you say about the points C, D, B?

(2) Consider two circles Γ_1 and Γ_2 which intersect at points A and B. Let MN be a line passing through A, with the point M on Γ_1 and the point N on Γ_2. As M and N vary on their respective circles, when does the segment $\bar{MN}$ reach maximum length?

(3) Given a triangle $\triangle ABC$ all of whose angles are less than $120°$, we construct equilateral triangles $\triangle ABC'$, $\triangle AB'C$ and $\triangle A'BC$ such that A' and A are on opposite sides of line BC, and similarly for the other pairs of vertices.
  a) **Warm-up:** Prove that $|AA'| = |BB'| = |CC'|$.
  b) Given a point M inside $\triangle ABC$, prove that $|MA| + |MB| + |MC| \geq |AA'|$.
  c) (*Fermat's problem*) Construct a point M_0 inside $\triangle ABC$, such that $|MA| + |MB| + |MC|$ reaches its minimum when $M = M_0$.

(4) **HW:** Given a square $ABCD$, find a network of segments of minimum length connecting the vertices of the square.

(5) **Warm-up:** Given a line l and two points A and B in plane on the same side of the line l, find the point M_0 on the line l such that the sum $|MA| + |MB|$ reaches its minimum when $M = M_0$.

(6) Given a triangle $\triangle ABC$ and three points D, E, F moving on its sides BC, CA and AB respectively, find the points D_0, E_0 and F_0 such that the perimeter $|DE| + |EF| + |FD|$ of $\triangle DEF$ reaches its minimum when $D = D_0$, $E = E_0$ and $F = F_0$.

GEOMETRY

Geometric Formulae and Geometric Loci

We consider $\triangle ABC$ with side lengths $|BC| = a, |AC| = b$ and $|AB| = c$.

(1) In $\triangle ABC$, consider the height $|AD| = h$. Let $t = |BD|$.
  a) Write down a system of two equations in the unknowns h and t, in terms of the known quantities a, b, c.
  b) Use the solutions of the system above to prove the following formulae:

The cosine formula: $\boxed{\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}}$ or, equivalently, $\boxed{b^2 = a^2 + c^2 - 2ac\cos B}$.

Heron's formula for area: $\boxed{[ABC] = \sqrt{s(s-a)(s-b)(s-c)}}$ where $s = \frac{a+b+c}{2}$.

(2) Let L be a point on the side BC of $\triangle ABC$. Prove *Stewart's formula*:

$$\boxed{AL^2 = b^2\frac{BL}{BC} + c^2\frac{CL}{BC} - a^2\frac{BL}{BC}\frac{CL}{BC}}.$$

If we denote $\frac{BL}{BC} = x$, the formula becomes: $\boxed{AL^2 = b^2x + c^2(1-x) - a^2x(1-x)}$.

Use this to calculate the length of the median AM.

(3) Let AK denote the angle bisector of $\widehat{BAC}$, where K lies on the side BC.
  a) Prove the following formula: $\boxed{\dfrac{AB}{AC} = \dfrac{KB}{KC}}$.
  b) Calculate the length of the angle bisector AK.

(4) Fix a segment AB of length c, three real numbers α, β, γ, and a fixed angle u. In each of the following cases, find the geometric loci of the points M satisfying the given conditions:
  a) $\widehat{AMB} = u$.
  b) $MA^2 + MB^2 =$ constant.
  c) $MA^2 - MB^2 =$ constant.
  d) $MA : MB =$ constant.
  e) $\alpha \cdot MA^2 + \beta \cdot MB^2 = \gamma \cdot c^2$.

(5) Fix a line l and a point P not passing through it. As a point A moves along l, find the geometric locus described by the a point A' on the segment AP satisfying

$$AP \cdot A'P = \text{ constant.}$$

(6) Consider a circle Γ and a fixed point D inside it. Fix a line l not passing through D. For every point A on l, we denote by BC the chord of Γ passing through D and perpendicular on AD. Find the geometric locus described by the orthocentre of $\triangle ABC$ as the point A moves along the line l.

Anca Mustata, School of Mathematical Sciences, UCC

E-mail address: `a.mustata@ucc.ie`