SIXTEENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 10 May 2003

2p.m.-5p.m.

Second Paper

6. Let T be a triangle of perimeter 2, and let a, b and c be the lengths of the sides of T.

   (a) Show that
$$abc + \frac{28}{27} \geq ab + bc + ac.$$

   (b) Show that
$$ab + bc + ac \geq abc + 1.$$

Solution. Observe that $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $0 \leq c \leq 1$. Thus
$$0 \leq (1-a)(1-b)(1-c).$$

Therefore
$$0 \leq 1 - (a+b+c) + (ab+bc+ac) - abc.$$

Now the first inequality follows immediately from the fact that $a+b+c = 2$. For the second inequality, observe that the AM-GM inequality implies that
$$(1-a)(1-b)(1-c) \leq \left(\frac{3-(a+b+c)}{3}\right)^3 = \frac{1}{27}.$$

Now the required inequality follows easily.

7. $ABCD$ is a quadrilateral. P is at the foot of the perpendicular from D to AB, Q is at the foot of the perpendicular from D to BC, R is at the foot of the perpendicular from B to AD and S is at the foot of the perpendicular from B to CD. Suppose that $\angle PSR = \angle SPQ$. Prove that $PR = SQ$.

Solution. Since the angles $\angle DPB, \angle DQB, \angle DRB$ and $\angle DSB$ are all right angles, the points P, Q, R and S lie on the circle with diameter DB. Thus, if the angles $\angle PSR$ and $\angle SPQ$ are equal, the chords that they stand on must be equal in length.

8. Find all solutions in integers x, y of the equation

$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

Solution. The equation can be written in the form

$$(y+1)^2 = (x^2 + 10x + 2)^2 + 2000 \qquad \ldots \qquad (1)$$

Let $u = y + 1, v = x^2 + 10x + 2$. Then (1) yields

$$u - v = c,\ u + v = d,$$

where c, d are integers satisfying $cd = 2000$. Then $v = (d - c)/2$, and thus

$$x^2 + 10x + 2 - (d - c)/2 = 0$$

must have an integer solution. Hence, from the quadratic equation formula,

$$92 + 2(d - c) \text{ must be a perfect square.} \qquad \ldots \qquad (2).$$

Thus, since $d - c$ must be even and $cd = 2000$, the only possibilities are

$(c, d) = (2, 1000), (-2, -1000), (4, 500), (-4, -500), (8, 250), (-8, -250), (10, 200), (-10, -200),$
$(20, 100), (-20, -100), (40, 50), (-40, -50),$

and those with (c, d) reversed.

The only value of $(d - c)$ satisfying (2) is $242 = 250 - 8 = -8 - (-250)$ and this leads to the solutions

$$x = -17, 7.$$

This forces $(y+1) = 129$, that is, $y = 128$, or $(y+1) = -129$, that is $y = -130$.

Thus the solutions are

$$(x, y) = (-17, 128), (7, 128), (-17, -130), (7, -130).$$

9. Let $a, b > 0$. Determine the largest number c such that

$$c \leq \max\left(ax + \frac{1}{ax}, bx + \frac{1}{bx}\right)$$

for all $x > 0$.

Solution. Let

$$g(t) = t + \frac{1}{t},\ t > 0,$$

and $f(x) = \max(g(ax), g(bx))$. (You might find it helpful at this stage to sketch the graphs of $g(ax)$ and $g(bx)$ for various a, b.) Since $g(t) \geq 2$, with equality if and only if $t = 1$, it's clear that $c \geq 2$, and this is the best we can do if $a = b$. So, suppose, without loss of generality, that $0 < a < b$. Now

$$g(bx) - g(ax) = bx - ax + \frac{1}{bx} - \frac{1}{ax} = (b-a)x(1 - \frac{1}{abx^2}).$$

Hence

$$f(x) = \begin{cases} g(bx), & \text{if } x \geq \frac{1}{\sqrt{ab}}, \\ g(ax), & \text{if } 0 < x \leq \frac{1}{\sqrt{ab}}. \end{cases}$$

But $g(s) \leq g(t)$ if $1 \leq s \leq t$ and $g(s) \geq g(t)$ if $0 < s \leq t \leq 1$. Hence $g(bx) \geq g(\sqrt{\frac{b}{a}})$ if $x \geq \frac{1}{\sqrt{ab}}$ and $g(ax) \geq g(\sqrt{\frac{b}{a}})$ if $0 < x \leq \frac{1}{\sqrt{ab}}$, whence

$$f(x) \geq g(\sqrt{\frac{b}{a}}) = \sqrt{\frac{b}{a}} + \sqrt{\frac{a}{b}},$$

with equality iff $x = \frac{1}{\sqrt{ab}}$. Thus

$$c = \sqrt{\frac{b}{a}} + \sqrt{\frac{a}{b}}.$$

10. (a) In how many ways can 1003 distinct integers be chosen from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10?

(b) Show that there are $(3(5151) + 7(1700))\, 101^7$ ways to choose 1002 distinct integers from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10.

Solution. We begin by proving a general result that we shall use repeatedly.

Lemma: Let m and n be positive integers and let $X = \{m, m+10, m+20, \ldots, m+10n\}$. Let k be a positive integer. If $k > \frac{n+2}{2}$ then it is impossible to choose k integers from the set X so that no two of the chosen integers differ by 10. If $k \leq \frac{n+2}{2}$, then there are $\binom{n-k+2}{k}$ ways to choose k integers from the set X so that no two of the chosen integers differ by 10.

Proof: Let $a_1 < a_2 < \cdots < a_k$ be elements of X and suppose that $a_j + 10 \neq a_{j+1}$ for all j. Let $b_j = a_j - 10(j-1)$. So $b_1 = a_1$, $b_2 = a_2 - 10$, $b_3 = a_3 - 20$ etc. Clearly

$$m \leq b_1 < b_2 < \cdots < b_k \leq m + 10(n - k + 1).$$

Observe that since $b_{j+1} \geq b_j + 10$ for each j, we must have $b_k \geq m + 10(k-1)$. Therefore $m + 10(k-1) \leq m + 10(n-k+1)$. Therefore, $k \leq \frac{n+2}{2}$. This proves that if $k > \frac{n+2}{2}$ then it is impossible to choose k elements of X so that no two of them differ by 10.

Moreover, the above construction provides a one to one correspondence between subsets of X containing k elements, no two of which differ by 10, and, subsets of $X' = \{m, m+10, \ldots, m+10(n-k+1)\}$ containing k elements. Now, X' contains $n-k+2$ elements, so if $k \leq \frac{n+2}{2}$, then X' has precisely $\binom{n-k+2}{k}$ subsets containing k elements. QED.

We can decompose the set $\{1, 2, \ldots, 2003\}$ into 10 disjoint subsets (think of them as the pigeonholes) in the following way. For $i = 1, 2, \ldots, 10$ let

$$A_i = \{m \in \mathbb{N} : 1 \leq m \leq 2003 \text{ and } m \equiv i \mod 10\}.$$

Thus, for example $A_1 = \{1, 11, 21, \ldots, 2001\}$ and A_1 has 201 elements. Similarly A_2 and A_3 have 201 elements. However, $A_4 = \{4, 14, \ldots, 1994\}$ has only 200 elements as do A_5, A_6, A_7, A_8, A_9 and A_{10}. Now, using the above Lemma we see that it is impossible to choose 102 elements from A_1 so that no two differ by 10. Similarly, it is impossible to 102 elements from either A_2 or A_3 in the required manner and it is impossible to choose 101 elements from A_4, A_5, A_6, A_7, A_8, A_9 or A_{10} in the required manner. So, by the (generalised) pigeonhole principle, if we are to choose 1003 integers in total we must choose precisely 101 from each of A_1, A_2 and A_3 and precisely 100 from each of $A_4, \ldots, A_{10}$. Thus, as a consequence of the Lemma, there are 101^7 ways in which we can choose 1003 integers from the set $\{1, 2, \ldots, 2003\}$ so that no two differ by 10.

If we have to choose a total of 1002 integers then there are two possibilities to consider. We could choose choose 100 integers from one of the sets A_1, A_2 or A_3, choose 101 from the other two and choose 100 from each of $A_4, \ldots, A_{10}$. There are $3\binom{102}{100}101^7$ ways to do this. The other possibility is that we choose 99 from one of sets $A_4, \ldots, A_{10}$, choose 100 from the remaining six of those sets and then choose 101 from each of the sets A_1, A_2 and A_3. There are $7\binom{102}{99}101^6$ ways to do this. Thus, the total number of possibilities is

$$3\binom{102}{100}101^7 + 7\binom{102}{99}101^6 = (3(5151) + 7(1700))101^7$$

as required.