SIXTEENTH IRISH MATHEMATICAL OLYMPIAD
Saturday, 10 May 2003
10a.m.-1p.m.
First Paper

1. Find all solutions in (not necessarily positive) integers of the equation

$$(m^2 + n)(m + n^2) = (m + n)^3.$$

Solution. Assume integers m, n satisfy the given equation. Then, expanding the terms in the equation, and simplifying, we see that

$$mn + m^2n^2 = 3m^2n + 3mn^2.$$

Equivalently,

$$mn(1 + mn - 3m - 3n) = 0.$$

Hence, either (i) $mn = 0$ or (ii) $1+mn-3m-3n = 0$. In case (i), the equation $mn = 0$ is satisfied only when either $m = 0$ and n is arbitrary, or $n = 0$ and m is arbitrary. Thus there are infinitely many *trivial* solutions, which can't be ignored!

In case (ii), we can rewrite the equation $1 + mn - 3m - 3n = 0$ in the form

$$(m - 3)(n - 3) = 8.$$

Hence $m - 3, n - 3$ are factors of 8, leading to the solution pairs

$$(11, 4),\ (7, 5),\ (-5, 2),\ (-1, 1),$$

and the corresponding ones when we interchange m and n.

2. P, Q, R and S are (distinct) points on a circle. PS is a diameter and QR is parallel to the diameter PS. PR and QS meet at A. Let O be the centre of the circle and let B be chosen so that the quadrilateral $POAB$ is a parallelogram. Prove that $BQ = BP$.

First Solution (from among many similar ones).

Since O is the midpoint of PS, it must be that OS is parallel and equal in length to AB. Thus $ABOS$ is a parallelogram. It follows that AS is parallel to OB. But SQ is perpendicular to PQ (angle in a semi-circle). So, AS, and hence OB is perpendicular to PQ. But O is equidistant from P and Q and hence lies on the perpendicular bisector of PQ. It follows that OB is the perpendicular bisector of PQ and hence B is equidistant from P and Q.

Second Solution—The Irish Method! There are many different ways of attacking geometry problems, though the purists would much prefer to see them done by synthetic methods. Here's a solution that uses coordinate geometry, which was favoured by some contestants. So, let P, Q, R, S be points on the unit circle $x^2 + y^2 = 1$ with coordinates $(-1, 0), (-c, s), (c, s), (1, 0)$, respectively, where $0 < c < 1$ and $s = \sqrt{1 - c^2}$. The lines PR, SQ have equations $y = m(x + 1), y = -m(x - 1)$, respectively, where

$$m = \frac{s}{1 + c} = \frac{\sqrt{1 - c^2}}{1 + c} = \sqrt{\frac{1 - c}{1 + c}}.$$

Hence A has coordinates $(0, m)$. It follows that $POAB$ is a rectangle and that $B = (-1, m)$. Using the distance formula,

$$\begin{aligned}
|BQ| &= \sqrt{(-1 - (-c))^2 + (m - s)^2} \\
&= \sqrt{(1 - c)^2 + s^2(\frac{1}{1 + c} - 1)^2} \\
&= \frac{\sqrt{(1 - c)^2(1 + c)^2 + s^2c^2}}{1 + c} \\
&= \frac{\sqrt{s^4 + s^2c^2}}{1 + c} \\
&= \frac{s}{1 + c}\sqrt{s^2 + c^2} \\
&= m \\
&= |BP|.
\end{aligned}$$

3. For each positive integer k, let a_k be the greatest integer not exceeding $\sqrt{k}$ and let b_k be the greatest integer not exceeding $\sqrt[3]{k}$. Calculate

$$\sum_{k=1}^{2003}(a_k - b_k).$$

Solution.

$$\sum_{k=1}^{2003}(a_k - b_k) = \sum_{k=1}^{2003} a_k - \sum_{k=1}^{2003} b_k.$$

We look at each sum separately.
Note that $44 \leq \sqrt{2003} < 45$, and, for $1 \leq n \leq 43$, $a_k = n$ implies $k \in \{n^2, n^2 + 1, n^2 + 2, ..., n^2 + 2n\}$, which has $2n + 1$ elements, and $a_k = 44$ implies $k \in \{1936, 1937, ..., 2003\}$, which has 68 elements. Thus

$$\sum_{k=1}^{2003} a_k = \sum_{n=1}^{43} n(2n + 1) + 68.44 = 58,806,$$

since

$$\sum_{n=1}^{m} n(2n + 1) = \frac{m(m + 1)(2m + 1)}{3} + \frac{m(m + 1)}{2} = \frac{m(m + 1)(4m + 5)}{6}.$$

Note next that $12 \le \sqrt[3]{2003} < 13$. Hence, for $1 \le n \le 11$, $b_k = n$ implies $k \in \{n^3, n^3+1, n^3+2, ..., n^3+3n^2+3n\}$, which has $3n^2+3n+1$ elements, and $b_k = 12$ implies $k \in \{1728, 1729, ..., 2003\}$, which has 276 elements.
Thus

$$\sum_{k=1}^{2003} b_k = \sum_{n=1}^{11} n(3n^2+3n+1) + 12.276 = 17,964,$$

since

$$\begin{aligned}\sum_{n=1}^{m} n(3n^2+3n+1) &= 3\left(\frac{m(m+1)}{2}\right)^2 + \frac{m(m+1)(2m+1)}{2} + \frac{m(m+1)}{2} \\ &= \frac{m(m+1)^2(3m+4)}{4}.\end{aligned}$$

Hence the sum is $58,806 - 17,964 = 40,842$.

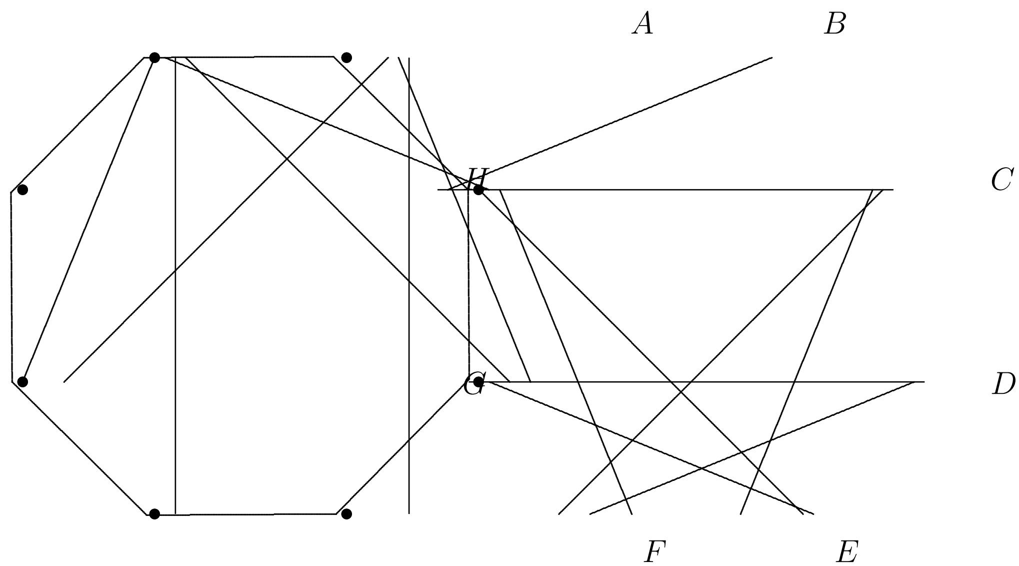
4. Eight players, Ann, Bob, Con, Dot, Eve, Fay, Guy and Hal compete in a chess tournament. No pair plays together more than once and there is no group of five people in which each one plays against all of the other four.

   (a) Write down an arrangement for a tournament of 24 games satisfying these conditions.

   (b) Show that it is impossible to have a tournament of more than 24 games satisfying these conditions.

Solution.

(a) We can satisfy the conditions and schedule the players so that each of them plays exactly 6 games. To see this, call the players A, B, C, D, E, F, G, H, and consider the following graph which models the desired scheduling. (Vertices denote players and two vertices I and J are adjacent (joined) if players I and J play together). It's very clear from this that any selection of five of the eight vertices includes a pair of opposite vertices which are non-adjacent.



(b) Suppose a tournament with 25 or more games is possible. Since each game has two players, this involves at least 50 instances of a person playing a game. Since no player can play more than 7 games, this means that at least two players play seven times each. Suppose that A and B each play seven games - this accounts for 13 games.

At least 12 games then involve neither A nor B. Of the remaining six players, either

(i) At least one player plays five games, *or*
(ii) Each plays in exactly 4 games.

We will show that neither of these is possible.

(i) Suppose C plays against all of D, E, F, G, H. This accounts for only five (of at least 12) games, so there is another game, e.g. D against E. Then each of A, B, C, D, E plays against each of the other four.
(ii) Assume we are in case (ii) and suppose C plays against D, E, F and G but not H. Then H plays also against D, E, F and G. This accounts for only 8 of at least 12 games. But now any game among the players D, E, F, G will create a forbidden group of five (A, B, C and two of D, E, F, G).

5. Show that there is no function f defined on the set of positive real numbers such that
$$f(y) > (y-x)(f(x))^2$$
for all x, y with $y > x > 0$.

Solution. Suppose that such a function f exists. Then, for each positive number x, $f(x)$ is a real number. Letting $y > 0$ be arbitrary and picking any $0 < x < y$, we see that $f(y) > 0$ for all $y > 0$. In particular, $f(1) > 0$. Taking $x = 1$ and $y \geq 1 + 4/f(1)^2$, we get
$$f(y) \geq \frac{4}{f(1)^2} \cdot f(1)^2 = 4 \quad \text{whenever} \quad y \geq z_0 := 1 + \frac{4}{(f(1))^2}.$$

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Thus P_0 is true, where the proposition P_n is defined as follows:

$$P_n : \qquad f(y) \geq 2^{n+2} \quad \text{whenever} \quad y \geq z_n := z_0 + \sum_{i=1}^{n} 2^{-i}.$$

Suppose P_k is true for a fixed integer $k \geq 0$. Taking $x = z_k$, and $y \geq z_k + 2^{-k-1} = z_{k+1}$ in the inequality, we get

$$f(y) > (y - z_k)(f(z_k))^2 \geq 2^{-k-1}(2^{k+2})^2 = 2^{k+3},$$

i.e., P_{k+1} is true. Inductively, P_n is true for all positive integers n. Now every z_n is less than $z_0 + 1$, so $f(z_0 + 1)$ would have to be larger than 2^{n+2} for all positive integers n, which is impossible.