



Maths Circles Ireland

18 Lesson Plans

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Foreword

For teachers and organisers, maths circles are the ideal class environment: no boring drills, no strict curriculum requirements, no mandatory attendance, no yawns or blank stares – all these replaced by an attentive and responsive audience that keeps coming back from week to week out of the sheer pleasure of working together. If this sounds too good to be true, it is because the members of the audience are selected – often, self-selected – by one criterion: a spark of curiosity about mathematics. It is then the purpose of the maths circles, to support these students, to nourish their investigative spirit, and to cultivate their talent.

Does this sound like too great a responsibility? In fact, the start of a maths circle is in itself a most important step. Youth find huge encouragement in the society of like-minded peers, their getting together once a week in an after school class becoming a celebration of their common interest. In this social context, Maths is no longer a skill to be embarrassed about, but the cool thing to do.

“Maths Circles are an excellent way for students to feel good about their maths ability and themselves, it gives them a like-minded social group and the freedom away from class to become excited again about maths.”

Celeste Quinlan, teacher

In my experience, soon the maths circle participants start building new friendships, challenging each other, learning from each other.

The other important step in running a maths circle is choosing a curriculum suitable for the audience and developing the actual activity/lesson plans. The purpose of this handbook is to provide the teachers relief from this daunting task. The handbook contains carefully selected activities which have been tried out in maths circles in the Cork area and have been found suitable for an audience at the beginning of the Junior Certificate cycle.

This handbook has benefited from the work of many members of the maths circles community around the Cork area, in particular of UCC students, graduates and lecturers who have drawn ideas from the wealth of resources available electronically or in print, and test-ran them in workshops and after-school classes. The whole project has benefited from sponsorship from the National Academy for Integration of

Research and Teaching through a 2011/2012 grant. By far the main workload in producing this book was taken by the author Ciarán Ó Conaill who, after running the Douglas Community School maths circle for almost a year, has then spent the Summer of 2012 reflecting on the experience, and writing up the material in the clear, well-structured and appealing format it has today.

The result is a set of lesson plans which are easy to implement and even easier to enjoy. With activities chosen so as to spur curiosity, dialogue, investigation, creativity, strategic thinking and self-confidence, they embody the philosophy of Maths Circles everywhere. At the same time, the unique way in which the activities are organised and presented exudes charm and local flavour: the lessons are dynamic, good-humoured and playful. While they address a nice variety of mathematical topics, they also imaginatively illustrate each topic through a multitude of contexts. They can carry students from the excitement of winning the Lotto to more sedate discussions of chess strategies. From the beginnings of magic tricks and number skills the students are seamlessly moved through a sequence of steps preparing them for proof-orientated problem-solving. All done in a light-hearted way that is bound to please and attract young participants.

Anca Mustata
Lecturer,
School of Mathematical Sciences,
University College Cork

Introduction

This is a sample course you could undertake with your own Maths Circle. All of the puzzles included here have already been used in Maths Circles around Cork. This book is designed to make it as easy as possible for you to run your own Maths Circle.

Each Lesson Plan contains a number of things:

- List of Resources needed.
- Outline of Activities.
- Instruction Sheets to be printed and given to students (if necessary).
- Solutions.

The first number of weeks use Puzzle Stations – this is where there are 3 puzzles for students to tackle. Each puzzle is set up, with any necessary resources, at its own table, with room for students to sit around it. Students are broken into 3 groups. Each group is assigned a puzzle station, where they work on the puzzle for around 10-15 minutes. The puzzles have extension questions, and are designed so that each student should be able to solve at least part of the basic puzzle, while the better students are kept busy with the extensions. The teacher circulates around the room, giving direction or hints where necessary. When the first 10-minute slot is over, students move, in their groups, to the next puzzle station. They stay there for another 10-15 minute slot, and then move onto the final station. At the end the teacher can do a quick summary of the puzzles, if necessary, or try to show a common theme between questions.

We found that students really enjoyed working in this format: they liked physically moving around, as well as having a different problem or set of problems every 15 minutes. If a student didn't like, or 'get', a particular question, they wouldn't be burdened with it for the whole class. At the same time, it allowed us to introduce topics in a number of guises, before returning to them later in the year, or in future years. Among other things, the puzzle circuits touch on: combinatorics; graph theory; arithmetic and geometric sequences; binary numbers; induction; and many other areas. When returning to these areas, whether in first year or later, we can use the puzzle station questions as examples.

In each of these lessons there is a take-home problem. We would go through one week's take-home problem as a warm-up at the beginning of the next class.

After Christmas, we got a bit more serious and settled down into more Whole-Class lessons, in a more traditional manner. At this stage the students were invested in the classes, and seemed more willing to invest time and energy in following lessons that might not be as appealing or diverting as the puzzle stations. A black/whiteboard is assumed as a resource in all of these lessons, and is not listed in each lesson plan.

Acknowledgements

This handbook would not have been written in its current form were it not for the following: the students at UCC who devised and developed a number of the puzzles that appear in this book, and who took part in Maths Circles workshops, sessions in schools, and the Maths Circle Open Day during 2011-2012 – without their enthusiasm this project would simply not be able to exist; in particular, to JP McCarthy, who worked out a number of the initial lesson plans in this book and ran the Maths Circle in Douglas Community School for the first number of weeks; the students at Douglas Community School, for their feedback and enthusiasm; Jennifer Murphy at Urseline Convent Secondary School in Blackrock for her resources; Ryan McCarthy, who developed solutions to a number of the lesson plans; Kieran Cooney, who took so many hours of his summer holidays to prepare lesson plans for 2nd/3rd year Junior Cycle students, thus making possible a follow-up programme for last year's Maths Circle students; and Anca Mustata, David Goulding, and Julie O'Donovan, who got the whole Maths Circles thing started, and made sure that I got this bit of it finished.

Ciarán Ó Conaill

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**Maths Circles
Ireland**

Week 1: Nim, Magic Squares, and Filling Jugs.

Introduction

This is a Puzzle Station lesson. Students begin with a multiplication warm-up problem, then work on three different challenges: a game (Nim); a familiar problem in a new context (Magic Squares); and a logic problem that could be used to look at mods (Filling Jugs).

Resources

- Students will need calculators for the Warm-Up Problem.¹
- A copy of each of the Puzzle Station question sheets for each student.
- One copy of the Magic Squares template, printed onto card and cut up.
- Something for Nim – the game is often played with matchsticks, but that's probably not the best idea in school!

Activities

Warm-Up Problem (5-10 mins): Calculator Trick

Take a three-digit number, e.g. 345, and write it out twice, to get 345,345.

Now, take this number, and then divide it by 7. Divide this answer by 11. Finally, divide this answer by 13. What do you get? (Answer: 345)

Students check that it works with any 3-digit number, then try to explain why.

Extension question (for those who see it straight away): Can you make a similar trick where someone starts with a 2-digit number, or a 4-digit number, or a 5-digit one?

Hint: Instead of going from 345,345 to 345 by division, think about what you'd need to multiply 345 by to get 345,345.

Puzzle Stations (10-15 mins each)

- **Nim: The Game.** Students play in pairs. A pile of 32 matchsticks is placed in the middle of the table, and players take turns removing one, two, or three sticks at a time. The aim of the game is to be the last person to remove a stick.

¹ We found the calculator on the computer useful for the Warm-Up Problem, because it can display a lot of digits. On your computer, go to *Start > Accessories > Calculator*, and when it opens go to *View > Scientific*.

- **Filling Jugs.** Students use unmarked jugs of 3, 5, and 8 litres to get exactly 4 litres of water. We got students to use their imagination rather than actual jugs, because of the mess involved, and the difficulty of getting jugs with only one capacity marked on them.
- **Magic Squares.** Students are given the numbers 1-9, cut out onto small squares, and asked to arrange them into a 3x3 square, so that each row, column, and diagonal adds to the same number.

Take-home Problem:

Add up all the numbers from 1 to 100 in your head.

Nim: The Game

This is a game for two players.

A bundle of 32 matchsticks is put in the middle of the table.

Players take turns to remove one, two, or three sticks at a time. The winner is the person who removes the last stick.



The question is: is it possible for one of the players to figure out a strategy, so that they always win?

Extensions

- The person going second can always win if there are 32 matches on the table. What other amount of sticks could you start with if you want the person going second to always win? Can you see a pattern here? Can you *explain* it?
- There are three very easy cases where the person going first can win – if there's just one, two, or three sticks on the table to start with! But what other starting amounts guarantee that the person going first can always win? And what should their strategy be?
- If the rules said you could take only one or two sticks at a time (instead of one, two, or three), does this change your strategy? Try to figure out, in this case, for what starting amounts can the person going first always win, and for what amounts can the person going second always win.
- What if you could take one, two, three, or four sticks at a time? Now, for what starting amounts can the first person always win, and for what amounts can the second person always win?

Filling Jugs

You are given 3 jugs, as illustrated:



8 litre jug
Full of water



5 litre jug
Empty



3 litre jug
Empty
(It's not far away, it's just small)

The aim is to get 4 litres of water in the 8 litre jug by pouring the water from one jug to another.

Because the jugs have no markings, you can only fill them up to the top – for example, you can't fill the 5 litre jug half way, and say that's 2.5 litres.

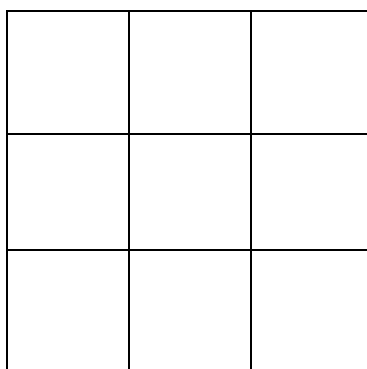
What's the least number of pours that you need to achieve this?

Extensions

- Could you get each whole number of litres in the 8 jug? I.e., could you get, in the 8 litre jug: 1 litre? 2 litres? 3 litres? 4 litres? 5 litres? 6 litres? 7 litres? I'm not saying that you can get each of them – some of them may be impossible!
- Suppose the jugs were 8 litres, 6 litres, and 4 litres, and only the 8 litre jug was full. Which of the amounts from the list above (1 litre, 2 litre, ..., 7 litres) could you get in the 8 litre jug? Explain, please!

Magic Squares

As you've met in primary school, the aim of a magic square is to fill in a 3x3 grid, such as the one below, with the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9. Each row, diagonal, and column must add up to the same number, called the **magic number**.



*A blank magic square, one of the true wonders of nature.
And a rabbit doing a magic trick.*

Believe it or not, there are 362,880 different ways you could fill in the square with the numbers from 1 to 9, but very few of them are magic squares.

Hints / Extensions:

- It turns out that 15 is the only magic number that works – but can you *explain* why? Remember, we need to use all of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.
- So, now that we know 15 is the magic number, we should ask ourselves: what number goes in the middle? We know it works with 5 – now try to fill in a magic square with 6 in the middle. Does it work? If not, why not?
- Now try to put 4 in the middle – does that work? Again, if it doesn't, explain why not.
- Finally, try to find what numbers *must* go in the corners. This is a good bit trickier, and requires a lot of patience!

Final Question: Can you fill in the numbers 1-9 in the magic square so that each row, column, and diagonal add up to *different* numbers?

Magic Squares Template

Print onto card (or laminate) and cut out the 9 individual squares.

1	2	3
4	5	6
7	8	9

Solutions

Warm-up Problem: Calculator Trick

345,345 is $345,000 + 345$. So to get 345,345 from 345 you need to multiply by 1,001.

The prime factors of 1,001 are $7 \times 11 \times 13$.

For the 2-digit number, you'd have to multiply by 101, e.g. $101 \times 56 = 5,656$.

However, as 101 is prime, the trick can't be disguised by breaking it into prime factors, which is why it's not such a cool trick with 2-digit numbers.

For the 4-digit number you would need $10,001 = 73 \times 137$.

For the 5-digit number you would need $100,001 = 11 \times 9091$.

Puzzle Stations

Nim: The Game. The second player always wins. If the first player removes x sticks, the second player removes $4 - x$ sticks. Because a player can only remove one, two, or three sticks, this is always possible. Thus the total number of sticks, after the second player takes their turn, will be (32), 28, 24, 20, ... , 4, 0. This means that the person going second wins, as long as the starting number is a multiple of four. If the starting number *isn't* a multiple of four, the person going first removes one, two, or three sticks so that the number left *is* a multiple of four, and then they (the person going first) will win.

If you could only take one or two sticks at a time, then the trick is to look at multiples of three. So the person going second will win if the starting number is a multiple of three, otherwise the person going first will win. In the case with 32 matchsticks, the person going first removes 2 matchsticks. Then, if the person going second removes x matchsticks, the person going first removes $3 - x$ matchsticks, and will win.

If you can take one, two, three, or four matchsticks, then it is the multiples of five that determine the winner. The second person will win if starting on a multiple of five, and the first person will win otherwise.

Filling Jugs. A table here may help:

Move	8 Litre Jug	5 Litre Jug	3 Litre Jug
Beginning.	8	0	0
Pour from 8 to 5.	3	5	0
Pour from 5 to 3	3	2	3
Pour from 3 to 8	6	2	0
Pour from 5 to 3	6	0	2
Pour from 8 to 5	1	5	2

Pour from 5 to 3	1	4	3
Pour from 3 to 8	4	4	0

In fact, you can get each of the values from 1 to 7 litres here.

All of them, except for 7, have been achieved in the table. To achieve 7, start at the last line. Pour the 5 litre jug into the 3 litre jug, to get {4, 1, 3}. Then pour the 3 litre jug into the 8 litre jug to get {7, 1, 0}.

With jugs of capacity 8, 6, and 4 litres, you can only get even results. This is because pouring from one jug to another can be thought of as adding and subtracting, and you can never get an odd number by adding and subtracting even ones. So you can get 2, 4, 6, and 8 litres. (You can think of it like having jugs of capacity 4, 3, and 2 litres, but where every litre is twice as big!)

Magic Squares. Note that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. So when you add the sums of the three rows, you get 45. If each row sums to the same number, this magic number must be $45/3 = 15$.

If you put the 4 in the middle, then you must put the 1 somewhere else. Now the 1 and the 4 lie in a line (a row, column, or diagonal) that adds to 15, so the third number in that line must be 10. But you don't have any 10, so you can't put 4 in the middle. (This can be explained as a nice proof by contradiction.) Similarly, you can't put 3 or 2 or 1 in the middle.

On the other hand, if you put 6 in the middle, then you must put the 9 somewhere else. Now the 6 and the 9 lie in a line that adds to 15, so the third number in that line must be 0. But you don't have a 0, so you can't put 6 in the middle. Similarly, you can't put 7 or 8 or 9 in the middle.

Finally, putting the odd numbers in the corners will lead to similar problems, so the even numbers must go in the corners.

Final Question: An example of an un-magic square is:

6	7	5
2	8	4
3	9	1

Take-Home Problem:

Put them in pairs:

- $1 + 100 = 101$
- $2 + 99 = 101$
- $3 + 98 = 101$
- ...
- $50 + 51 = 101$

As there are fifty pairs, the answer is $101 + 101 + \dots + 101$, fifty times.

In other words, $50 \times 101 = 5050$.

Week 2: Matchstick Mania, Dastardly Diagrams, and a Crazy Fly.

Introduction

This is a Puzzle Station lesson. Students meet some physical puzzles (Matchstick Mania), look at what kind of diagrams you can draw without lifting your pen from the paper (Dastardly Diagrams), and a speed/distance/time problem that can be solved in a very easy, or a very difficult way (Crazy Fly). The last problem could be used to introduce the idea that an infinite sequence can add to a finite sum. It also has the Barber Paradox as an extra question.

Resources

- Something for Matchstick Mania – again, please don't use real matchsticks!
- Students may need calculators for the Crazy Fly (if they do it the hard way!).
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Matchstick Mania.** Students are each given a worksheet and 16 sticks. They should work through the problems, together or independently. Each problem involves moving a certain number of matchsticks in a given shape to create a new shape.
- **Dastardly Diagrams.** Students are each given a worksheet. The aim is to begin to classify what kind of shapes can be drawn without lifting your pen from the paper and without tracing over any given edge more than once.
- **The Crazy Fly.** Students are each given a worksheet. There are two problems – one involves a crazy fly, and the other is the Barber paradox. Part of the aim of this is to show students that not all maths problems have answers.

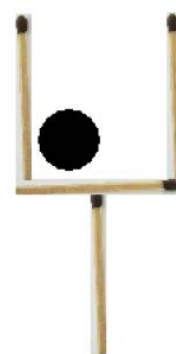
Take-home Problem:

How many squares are there on a chessboard?

Matchstick Mania

Challenge 1:

Get the cherry outside the glass on the right by moving exactly 2 matchsticks. You must end up with an intact glass, exactly the same size as the one you started with!

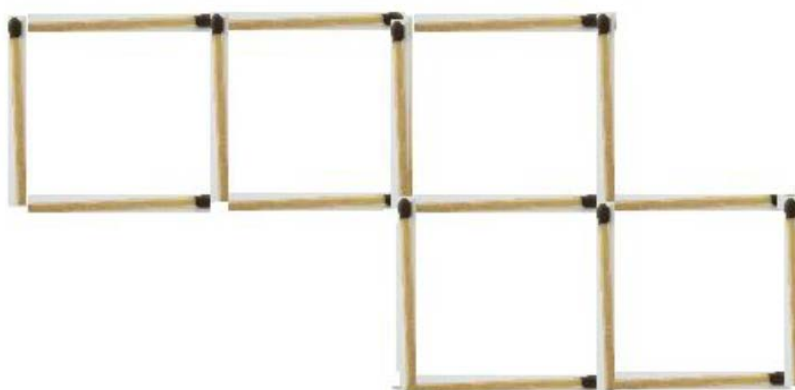


Challenge 2:

Turn these 4 squares of side length 1 matchstick into three squares of side length 1, by moving exactly 4 matchsticks.

Challenge 3:

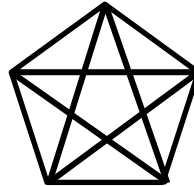
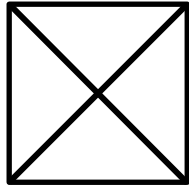
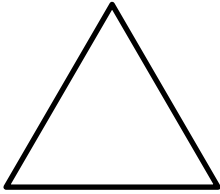
Turn these 5 squares of side length 1 into 4 squares of side length 1 by moving exactly 2 matchsticks.



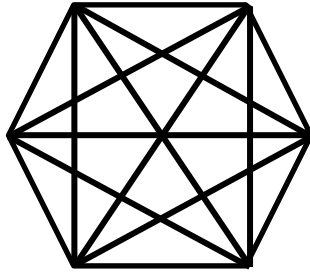
Dastardly Diagrams

The question in all of these problems is: can you draw the shape without lifting your pen off the paper, and without tracing any edge more than once?

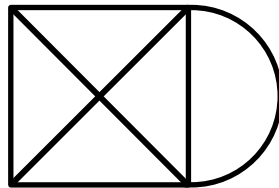
Problem 1: Can you draw the following shapes without lifting your pen?



Problem 2: Guess whether you can draw the following shapes without lifting your pen. Then check your guess and see if you were right!

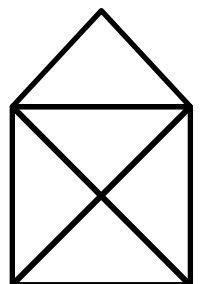
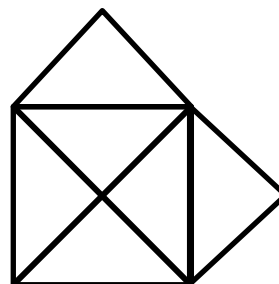
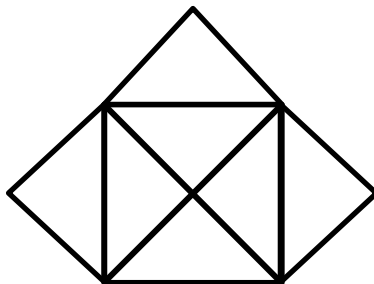
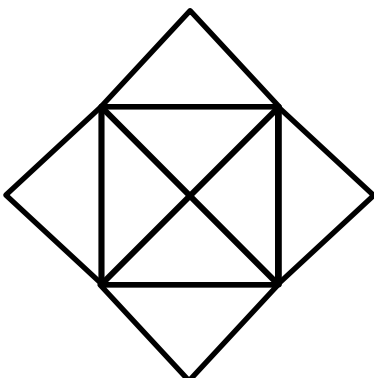


Problem 3: You *can* draw the square below, which I like the square above but with extra edge. In this case, which points can you start from, and draw the shape? And which points can you *not* start from? Can you explain why?

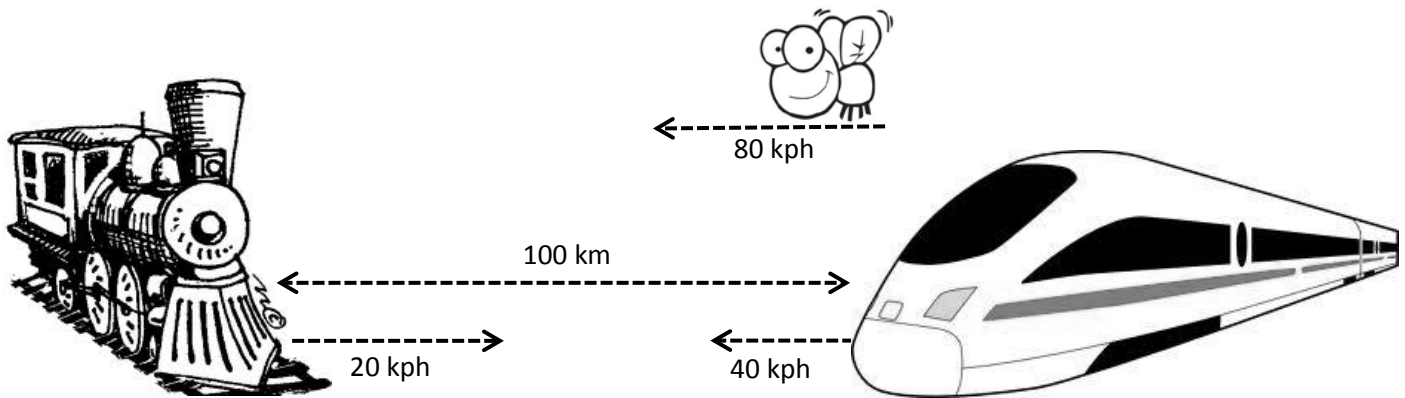


Problem 4: For each of these, decide which heading they fall under:

- (a) You can't draw them without lifting your pen.
- (b) You can draw them, but you must start at a particular point.
- (c) You can draw them, starting at any point.



The Crazy Zig-Zagging Fly



A fly at the front of the train travelling at 40 kph takes off flying at 80 kph until it reaches the front of the second train. It then immediately turns around and flies back towards the first train. It turns around again and flies back towards the other train.

As the trains get closer and closer together, the fly zig-zags back and forth over shorter and shorter distances, until the trains touch, and the fly is killed.

How far does the fly fly?

Extra Question.

Most people in a town shave themselves.

The barber shaves anyone who doesn't shave themselves, and no-one else.

So, who shaves the barber?

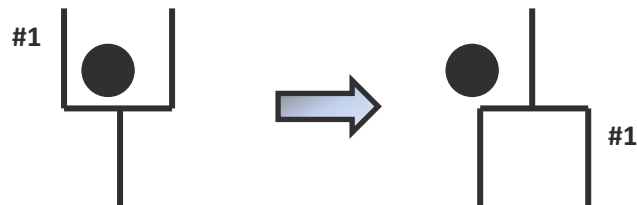


Solutions

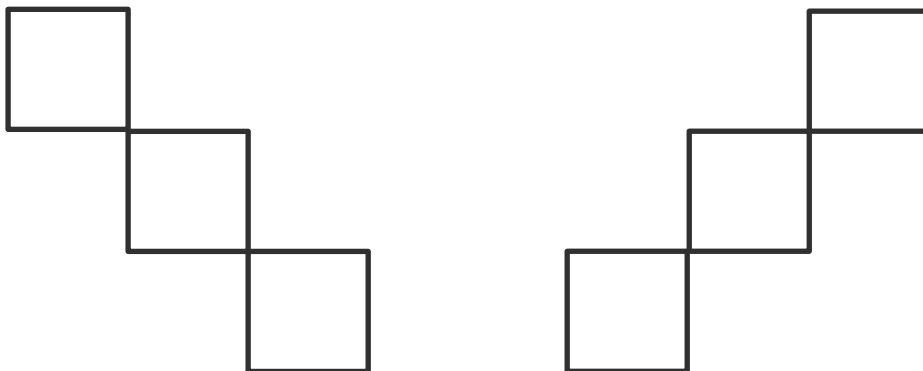
Puzzle Stations

Matchstick Mania.

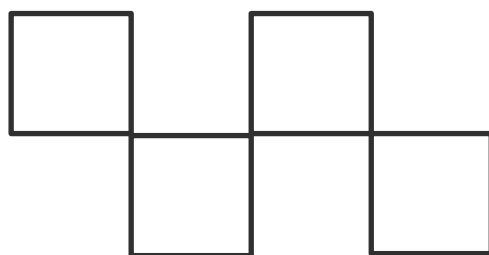
Challenge 1: Move the matchsticks so that the glass is the same shape, but upside down. To do this, move matchstick #1 as shown, and slide the base of the glass (the horizontal matchstick) to the right.



Challenge 2: You have 12 matchsticks, and want to get just 3 squares. This means that none of the squares can share a side (at the moment, each square is sharing two sides with other squares). So move the matchsticks to get a diagonal arrangement – either of these will do:



Challenge 3: Again, you have 16 matchsticks, and want to create 4 squares, so you'll need to make sure that they don't share any edges. To do this, arrange them like this:



The Crazy Zig-Zagging Fly. One way to do this is to find out:

- How long the fly travels before it meets the slow train: The fly travels 4 times faster than the slow train, so the distance covered will be broken in the ratio 4:1. As they must cover 100km to meet, the fly travels 80km.
- How long it travels before it meets the fast train again: It travels twice as fast as the fast train, so if it travelled 80km initially, the fast train travelled 40km. So when it turns back, it has 40km to cover. This time, the distance covered will be in the ratio 2:1. So the fly will travel $\frac{2}{3} \times 40 = 26\frac{2}{3}$ km.
- And so on, ad infinitum.

A more straightforward way to do the problem is to see that, between them, the trains travel $40 + 20 = 60$ km in an hour. So it will take them $\frac{100}{60} = \frac{5}{3}$ hours before they meet. As the fly is going at 80 km/hr, in this time it will travel $\frac{5}{3} \times 80 = 133\frac{1}{3}$ km.

Extra Question.

This is a famous paradox, or contradiction, related to Russell's paradox:

- Suppose the barber doesn't shave himself. Then he must be shaved by the barber, because the barber shaves anyone who doesn't shave themselves. So he must shave himself – a contradiction!
- So the barber must shave himself. But the barber only shaves people who *don't* shave themselves. So if he shaves himself, then he can't shave himself – another contradiction!

Dastardly Diagrams. The trick here is to spot that it's all about the *degree* of the corners – that is, how many edges meet at a given corner. (This branch of maths – graph theory – was invented by Leonhard Euler in response to a question about the Bridges of Königsberg, which is definitely worth a Google!) We'll explain the general theory, then very quickly go through the questions asked. It's probably as well to let the students play around with the diagrams here, and wait to introduce the theory in a later lesson.

Suppose I'm drawing one of these shapes. We can split the corners into three classes (though the first and second might be the same corner):

- The corner I start at.
- The corner I end at.
- The "middling" corners.

Let's look at the middling corners first. At these corners, I will draw the edges that connect into it two at a time – I draw one edge when I enter the corner, and another

edge when I exit it. So a middling corner must have an even number of edges, i.e. an even degree.

Now suppose I want to start and end at the same corner. Call this corner C_1 . Two of the edges at C_1 will be the first and last edges that I draw. The rest of the edges can be treated as in the case of the middling corners, so again, C_1 must be of even degree.

On the other hand, suppose that I want to start at C_1 and end at a different corner, C_2 . Once I've drawn the first edge out from C_1 , I can then treat it as a middling corner. So this means that C_1 must have an odd degree. Similarly, once we eliminate the final edge that I draw into C_2 , then we can treat C_2 as a middling corner. This means that C_2 must have odd degree.

Finally, suppose that we have more than two corners of odd degree. We can use at most two of them as starting and ending corners, which means that we have at least one middling corner with odd degree. This is impossible, so we cannot draw this graph.

(It's also worth noting that the number of corners with odd degree must be even, e.g. we can't have 1 corner of odd degree, or 3 corners, or 5. To see this, observe that each edge meets two corners, so that if you add the degree of each of the corners, you get double the number of edges, i.e. an even number. So when you add the degree of each of the corners, there must be an even number of corners with odd degree, in order to give an even number as an answer.)

Problem 1. The triangle and the pentagon can be drawn – the square can't. Each corner in the triangle has degree 2 (i.e. 2 edges meet at each corner), and each corner in the pentagon has degree 4. On the other hand, the square has four corners, each of degree 3, which is odd. So we won't be able to draw that.

Problem 2. You can't. Each of the outside corners of the shape has degree 5, which is odd.

Problem 3. You must start (and end) at the corners with odd degree, i.e. the top left or bottom left. You can't draw it if you start from another corner, as you can't use the corners with odd degree as middling corners.

Problem 4. The first one is (a), as it has four corners, all with degree 5. The rest are all (b), as each of them has two corners of odd degree. In each case, you must start (and end) at a corner with odd degree. Note that the degrees of the odd corners don't have to be the same – the third diagram has one corner of degree 3, and one of degree 5.

Take-home Problem

There are:

- $8^2 = 64$ small 1×1 squares.
- $7^2 = 49$ slightly bigger 2×2 squares.
- $6^2 = 36$ slightly bigger again 3×3 squares.
- ...
- $1^2 = 1$ big 8×8 square.

So altogether there are $1^2 + 2^2 + \dots + 7^2 + 8^2 = 204$ squares on a regular 8×8 chessboard.

Week 3: Mastermind, Mobius Strips, and Bothersome Brainteasers.

Introduction

This is a Puzzle Station lesson. Students play Mastermind (which we'll return to in Combinatorics), make Mobius Strips, and solve some Brainteasers.

Resources

- Two or three Mastermind games, depending on numbers. If you don't have Mastermind games, you can simply use paper – the codemaker writes her code on a piece of paper that she keeps covered; the codebreaker writes her guesses on a sheet of paper; and the codemaker writes feedback next to each guess. Just decide in advance what colours are being used!
- Sheets of plain paper and a few scissors for the Mobius Strips.
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Mastermind.** Students play the game in pairs – one person creates a code, using coloured pegs, and the other has to guess the code, in as few goes as possible.
- **Mobius Strips.** Students make Mobius strips, with one or more turns, and explore some of their bizarre properties.
- **Brainteasers.** Students solve a number of brainteasers – some are logic puzzles, some more maths puzzles.

Take-home Problem:

I have a drawer with 8 green socks, 9 yellow socks, 10 orange socks, and 11 blue socks, all for my dog. If I reach into the drawer without looking and start picking out socks, how many do I need to pick to be sure that I can put 4 socks of a matching colour on my dog? (I don't care what the colour is.)

Mastermind

This is a game for two players, a codemaker and a codebreaker. The codemaker makes a code, which consists of four coloured pegs in a row. The codebreaker must try to figure out the code, using as few guesses as possible.



If the codebreaker figures out the code, then they win.

If the codebreaker doesn't manage to figure it out, then the codemaker wins!

Directions, in Detail:

1. One player becomes the codemaker, so the others are the codebreakers.
2. The codemaker chooses a pattern of four code pegs. Initially, you could restrict the game so that repeated colours are not allowed. If students are very fast to pick up on the game, you could then relax this rule. The chosen pattern is placed in the four holes covered by the shield, visible to the codemaker but not to the codebreaker (circled in picture above).
3. On each go, the codebreaker guesses a code, by placing 4 pegs in one of the rows on the board.
4. The codemaker then places up to four small red or white pegs in the holes at the side of the board:
 - If a peg is the correct colour, and in the correct position, a small red peg is placed.
 - If the peg is the correct colour, but in the wrong place, a small white peg is placed.
 - If a peg is the wrong colour, no small peg is placed.
5. Once the codebreaker has done this, the codebreaker can take another guess at the code, until they run out guesses.

Example:

	Code:	Feedback (small pegs)
Code	B R G B	
Guess 1	B Y B P	red (first B), white (second B)
Guess 2	P Y G G	red (first G)
Guess 3	B B R G	red (first B), white (second B), white (R), white (G)

B: Blue. R: Red. G: Green. Y: Yellow. P: Purple. O: Orange.

Mobius Strips

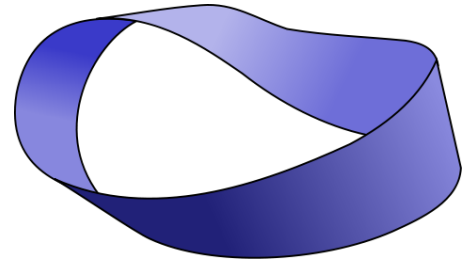
The Mobius Strip is a very simple-looking shape with some very bizarre properties! It was invented in 1858 by Augustus Ferdinand Mobius, a German astronomer and mathematician.

We'll start by making an ordinary loop of paper. Take a long strip of paper, and tape the two ends together to make a loop. We'll ask three way-too-simple-looking questions:

1. How many edges does the loop have? Check by running your finger along the edge(s), until you get back to where you started.
2. How many sides does the loop have? Check by drawing a line down the middle of the strip of paper – continue drawing until you get back to where you started.
3. What would happen if you were to cut along this line?

Now to make a Mobius Strip. Take a long strip of paper, but before you tape the ends together, give one of them a half-twist. When you do, you should get a twisted band, like the one on the right.

Now try answering the same three questions for the Mobius Strip:



4. How many edges does it have? Check with your finger.
5. How many sides does it have? Check by drawing a line.
6. What do you think will happen if you cut along this line? Check by cutting!

Further Investigations:

7. Make another Mobius strip – make this one nice and wide. Try to cut this Mobius Strip, starting one third of the way in from one edge. The cut will meet itself eventually, after a few laps of your Mobius Strip. What do you think you will be left with when you do this? Check!
8. Take another strip of paper, and this time do a full twist (instead of a half-twist) when you join the ends. How many edges and sides does this shape have? What will happen if you cut it down the centre? Check!
9. Can you think of any practical uses for the Mobius strip?

Bothersome Brainteasers



1. Suppose you want to cook an egg for exactly three minutes. You have only got a five-minute hour glass timer and a two-minute hour glass timer. Using these two timers, how can you boil the egg for exactly three minutes?
2. You have nine marbles. Out of the nine marbles there is one marble that weighs slightly heavier than the rest. The rest of the marbles all weigh the same. How would you find the heaviest marble, if you can only weigh them two times using a pair of scales? (A scales is like a see-saw: it will tell you which side is heavier.)
3. Two boys and a man need to cross a river. They only have a canoe. It will hold only the man OR the two boy's weight. How can they all get across safely?
4. Four cards are laid out on a table. Each card has a letter on one side and a number on the other side. The sides of the cards that we can see read:



An unreliable source told us that whenever a "7" is on one side of a card, an "S" is on the other side. The task of this puzzle, is to check if this unreliable source is telling the truth. However, you can only turn over two cards. So, which two cards should you turn over?

Hint: It might help if you wrote out the four characters onto four pieces of paper and tried doing it.

5. You face two guards: A knight and a knave. The knight will always tell the truth and the knave will always lie, and you do not know which is which. You must find out which one is the knight and which one is the knave, and are only allowed to ask one question to one of the guards.

What question should you ask?

6. In the basement there are three light switches in the "off" position. Each switch controls one of three light bulbs on the floor above. You may move any of the switches, but may only go upstairs one time. How can you determine which switch controls each light?



Solutions

Puzzle Stations

Mastermind.

There are a number of algorithms that will guarantee a win in 6 guesses or less – simply Google “winning strategies for mastermind.” However, these are best suited to computers, and not people playing the game! Make sure, as you’re going along, that any guess you make is consistent with the answers you’ve gotten from the codemaker up to that point. An interesting article is *Yet Another Mastermind Strategy* by Barteld Kooi – it has a good overview of a number of different strategies, and explains, among other things, why a first guess of RRGG would be preferable to RRRR, or even RGBY. If you’re looking for more details on optimal strategies, it’s a good place to start.

Mobius Strips.

1. The loop has two edges.
2. The loop has two sides – no surprises yet!
3. You would cut the loop into two smaller loops.
4. The Mobius strip only has 1 edge! If you were start at a particular point on the edge and move around the edge with a black marker, you would colour the *entire* edge of the strip before getting back to where you started – there wouldn’t be any white edge left.
5. Similarly, the Mobius strip has only one side.
6. You will cut it into one larger Mobius Strip, which is half the width and twice the length.
7. Amazingly, you will be left with 2 interlocking Mobius strips, one the same length as the original, and the other twice the length.
8. This shape has 2 edges and 2 sides. (In fact, a Mobius strip with an odd number of half-twists will have 1 edge and 1 side, while one with an even number of half-twists will have 2 edges and 2 sides.) If you cut this strip down the middle, you will produce 2 interlocking Mobius strips!
9. Impressing your friends could be one. More practically, typewriter ribbons and conveyor belts are sometimes in the form of Mobius strips – this is because a regular conveyor belt will get very worn on the side facing outwards, and not very worn on the side facing in. A conveyor belt in the shape of a Mobius strip will wear evenly on both sides – more correctly, it only has 1 side to get worn!

Brainteasers.

1. Turn over both glass timers simultaneously. When all the sand has drained from the 2-minute timer, put the egg in the boiling water. Once all the sand has drained from the 5-minute timer, 3 minutes will have elapsed.
2. First of all place 2 groups of 3 marbles on the scales, one in each pan. It doesn't matter how the groups are selected. Now, one of 2 things will happen:
One group of 3 is heavier: If this happens, select the heavier group of 3. From it, pick out 2 marbles, and put them on the scales, 1 in each pan. If 1 of the marbles is heavier, then this is your odd one out. If they are the same weight, then the marble you omitted is the heavy one.
Both groups of 3 are the same weight: In this case, the heavy marble is in the group of 3 that you omitted. Now take that group, pick out 2 marbles, and put them on the scales, 1 in each pan. If 1 of the marbles is heavier, then this is your odd one out. If they are the same weight, then the marble you omitted is the heavy one.
3.
 - (i) The 2 boys row across.
 - (ii) 1 boy rows back (the other stays).
 - (iii) The man rows across.
 - (iv) The second boy rows back to the first boy.
 - (v) Both boys row across the river.
4. This is counterintuitive. The solution is to turn over the 7 card and the J card. It's fairly clear that we need to check the other side of the 7, but what about the J – should we not check the S instead? Well, suppose the S had a 6 on the other side. The rule says: "Whenever there's a 7 on one side, there's an S on the other." It doesn't tell us that S should be on the other side *only when* 7 is on the first side – so we could have a 6 on the first side and an S on the other side without breaking the rule. On the other hand, if there's a 7 on the other side of the J, then the rule is definitely false – we would then have a 7 on one side but wouldn't have an S on the other. So the J is the one we need to check.
The question shows how a rule can be falsified, but can't be proven, by a single example.
5. Basically, any statement that everyone involved will know the answer to will do here. For instance, you could ask: "Are you both Knights?" to either of the guards. The answer is no, so the Knight will always say no (telling the truth) and the Knave will always say yes (as he lies). Similarly, asking "what's $2 + 2$ " would do as well, assuming that both the Knight and Knave progressed that far in their mathematical education.
6.
 - (i) Turn on 1 switch for 5 minutes (call this bulb A).

- (ii) Turn off the first switch, then turn on another one and leave it on (bulb B).
- (iii) Without touching the last switch, enter the room. Bulb A will be off and warm. Bulb B will be still lighting. Bulb C will be cold.

Take-home Problem

This is an example of the Pigeonhole Principle. The worst-case scenario here is that I pick out 3 socks of each colour, giving me $3 \times 4 = 12$ socks without 4 matching.

Now, no matter what sock I pick out next, I must have 4 of the same colour. Thus 13 is the least number I must pick to be sure that I have a quadruple.

Week 4: The Flummoxed Flea, the Money Maximising Muddle, and the Sierpinski Sponge.

Introduction

This is a Puzzle Station lesson, organised around a common theme: sequences and series. The problems introduce, in a casual way, both arithmetic and geometric sequences, and at both finite and infinite sequences.

Resources

- Calculators, for all questions. The calculator on the laptop would be useful.
- Students might find box paper useful for the Sierpinski Sponge.
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **The Flummoxed Flea.** A flea hops half way across a table, then half the remaining distance, then half the remaining distance, etc. Introduces the notion of an infinite sequence, and the possibility that the sum of an infinite number of terms might be a whole number.
- **The Money Maximising Muddle.** Students look at three different money-making schemes – one is a constant sequence, one is arithmetic, and one is geometric (it doubles each day). The aim is to find the sum of the first 49 terms of each sequence, and see which is biggest.
- **The Sierpinski Sponge.** This involves finding the area of the Sierpinski Carpet, a simple fractal created using squares. Again an infinite sequence, this explores the possibility of a sequence converging to 0, so it is linked very closely to the Flummoxed Flea.

Take-home Problem

How many factors does $2^{10} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ have?

The Flummoxed Flea

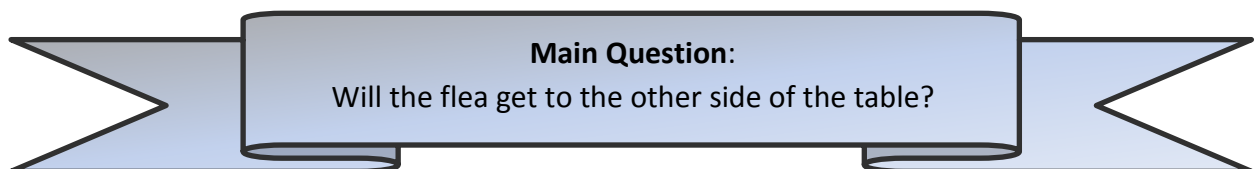
A flea is jumping across a table that is 1 metre wide to get from one yummy snack to another. He starts at the border of the table, and is trying to cross the whole way over to the other border.

However, due to malnutrition, the flea's jumps get shorter and shorter, as follows:

- In the first jump, the flea jumps half way, i.e. 50cm.
- In the second jump, the flea jumps half the previous jump, i.e. 25cm.
- In the third jump, the flea jumps half of the previous jump, i.e. 12.5cm.
- Etc.

1. Work out how far along the flea is after each of the first five jumps – show this on a diagram.

The main question now is:



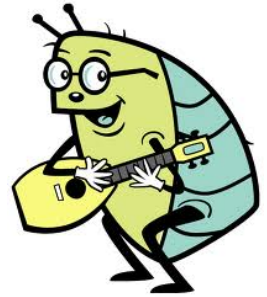
Some questions to think of on the way:

2. Will the flea ever get past 90 cm? If so, after how many jumps?
3. Will the flea ever get past 99 cm? If so, after how many jumps?
4. Will the flea ever get to the end of the table? If so, after how many jumps?

It can help to think in the following way:

5. What fraction of the table does the flea have left to get across after:
 - i. The first jump?
 - ii. The second jump?
 - iii. The third jump?
 - iv. The sixth jump?
 - v. The tenth jump?
 - vi. The hundredth jump? (Don't try to give this answer as a decimal or a fraction. Instead, give it as a fraction to a (big!) power.)

Now try to answer the Main Question again – will the flea ever cross the whole table?



The Money Maximising Muddle

You are offered a job for your summer hols – it's for 7 weeks, 7 days a week, and you can choose any one of the four following payment schemes:



Scheme 1: You'll be paid €2,000 per day for the 7 weeks.

Scheme 2: You'll be paid €100 for the first day, €200 for the second day, €300 for the third day, and so on.

So each day, you'll be paid €100 more than the day before.

Scheme 3: You'll be paid 1 cent the first day, 2 cent the second day, 4 cent the third day, and so on.

So each day, you'll be paid double what you were paid the day before.

Scheme 4: You'll be paid 1 cent on the first day, and on each of the following days you'll get paid as much as in all the previous days put together, plus 1 cent.

Work out how much each scheme will end up paying you, and decide which scheme is the best!

Try to find a quick way of getting the total pay for Schemes 2, 3, and 4 – in each case, there's a pattern going on, although the pattern may not be the same for each one.

The Sierpinski Sponge

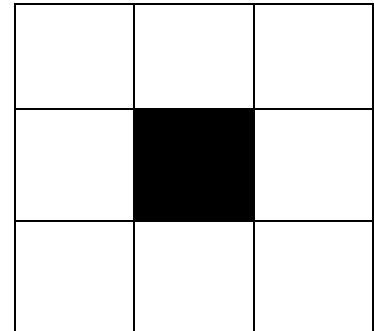
Technically, this is the Sierpinski Carpet, but there's no alliteration there, so we changed it. Anyway, if you do this in 3 dimensions, instead of 2, you get the Sierpinski Sponge – just Google it!

So, we're going to start off with a square – draw a nice big square on graph paper. For reasons that will become clear in a while, make it 27 boxes wide and 27 boxes high.

Now follow these simple steps:

1. Break your square up into 9 equal squares, as on the right, and colour in the middle square.

Question: How much of the square have you removed?
So how much of the square is left?



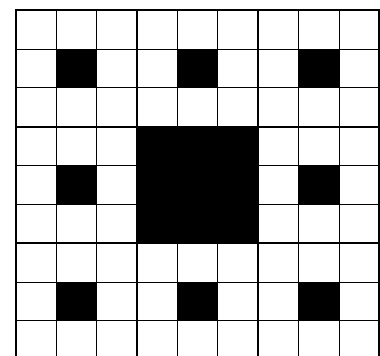
Carpet 1

2. Now take each of the 8 smaller squares that are left, divide each of *these* into 9 equal squares, and colour in the middle square in each.

Question: How much of each square have you removed?

So how much of *Carpet 1* have you removed? And how much is left?

Further Question: How much of the original square is now left? Can you work out your answer without counting little squares? If you can, check your answer using the little squares.



Carpet 2

3. Now take each of the 64 small squares, divide each of *these* into 9 equal squares, and colour in the middle square in each.

Question: How much of each square have you removed? So how much of *Carpet 2* have you removed? And how much is left?

Further Question: How much of the original square is now left?



The BIG Question:

How much of the original square would be left if you did this process:

- One more time?
- Ten more times?
- A hundred more times?
- Forever??

Again, no. After n jumps he will have $\frac{1}{2^n}$ left, which is always a positive distance. This distance will, however, get very very very small as n increases. In fact, for any tiny distance you care to pick, this number will eventually get smaller than that. So in this case we say that 0 is the limit of the series $\frac{1}{2^n}$ as n goes to infinity.

The Money Maximising Muddle.

Scheme 1: The total payment is $2000 \times 49 = 98000$.

Scheme 2: You get paid €100 the 1st day, €200 the 2nd, €300 the 3rd, and so on, up to €4900 on the 49th day. To add all these up, group them:

- $100 + 4900 = 5000$
- $200 + 4800 = 5000$
- $300 + 4700 = 5000$
- ...
- $2400 + 2600 = 5000$

There are 24 pairs here, each adding to €5000 – this gives a total of $24 \times 5000 = 120000$. Also, the 25th day doesn't have a day to pair off with, so we must add in an extra €2500, to get a total of €122500.

Scheme 3: Although it might not seem like much at first, starting with 1 cent and doubling each day for 49 days will generate a *huge* amount. Take a look at the following table:

Day	Amount on day (cent)	Amount so far (cent)
1	$1 = 2^0$	1
2	$2 = 2^1$	3
3	$4 = 2^2$	7
4	$8 = 2^3$	15
5	$16 = 2^4$	31

On the n th day, you are paid 2^{n-1} cent, or $\frac{2^{n-1}}{100}$ euro.

Also, your total on day n is given by $2^n - 1$ cent, or $\frac{2^n - 1}{100}$ euro. One way to see this is to observe that, for instance, $1 + 2 + 2^2 = 2^3 - 1$. Adding 2^3 to this gives: $(1 + 2 + 2^2) + 2^3 = (2^3 - 1) + 2^3 = 2^3 + 2^3 - 1 = 2 \times 2^3 - 1 = 2^4 - 1$.

A similar (inductive) argument will work for $1 + 2 + 2^2 + \dots + 2^n$. Or, if you would prefer to avoid introducing induction in a formal manner, you could simply indicate that, because it works for the sum up to 2^3 , it will work in a similar manner for the sum to 2^4 , or 2^5 , or, indeed, 2^n .

Alternatively, you could take the total earnings up to the end of day $n + 1$ and add 1 to it, to get: $1 + 1 + 2 + 2^2 + \dots + 2^n$.

Then $1 + 1 = 2$, so this is: $2 + 2 + 2^2 + 2^3 + \dots + 2^n$.

But $2 + 2 = 4 = 2^2$, so you get: $2^2 + 2^2 + 2^3 + \dots + 2^n$.

Now $2^2 + 2^2 = 2 \times 2^2 = 2^3$, so you get: $2^3 + 2^3 + \dots + 2^n$.

You can continue this until you get $2^n + 2^n = 2 \times 2^n = 2^{n+1}$.

Thus $1 + 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1}$, so $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$.

Whichever way you look at it, by the end of the 49th day you would have a total of $\frac{2^{49}-1}{100}$ euro, a pretty sizeable amount of cash!

Scheme 4: Let's take a look at how much you get paid each day:

Day	Amount on day (cent)	Amount so far (cent)
1	1	1
2	$1 + 1 = 2$	3
3	$3 + 1 = 4$	7
4	$7 + 1 = 8$	15
5	$15 + 1 = 16$	31

This is exactly the same as the previous scheme – the question is, why is this?

Again, we could take an inductive approach – observe that, on the first day, the amount I've been paid so far is $2^1 - 1$.

1 more than this is 2^1 , so I'm paid this on day 2. So my cumulative total is now $2^1 - 1 + 2^1 = 2 \times 2^1 - 1 = 2^2 - 1$. A similar argument can be made for days 3, 4, 5, and so on. Indeed, the same argument would work for day $n + 1$, presuming the result for day n , though again you might wish to avoid formalising the inductive step.

The Sierpinski Sponge.

1. You have removed $\frac{1}{9}$ of the square, so there is $\frac{8}{9}$ left.

2. **Question.** You've removed $\frac{1}{9}$ of each *smaller* square, so you have removed $\frac{1}{9}$ of *Carpet 1*. This means that you're left with $\frac{8}{9}$ of *Carpet 1*.
- Further Question.** There is now $\frac{8}{9}$ of $\frac{8}{9}$ left, i.e. $\frac{8}{9} \times \frac{8}{9} = \frac{64}{81}$ left. You can check this by counting the little squares.
3. **Question.** Again you've removed $\frac{1}{9}$ of each *smaller* square, so you have removed $\frac{1}{9}$ of *Carpet 2*. This means that you're left with $\frac{8}{9}$ of *Carpet 2*.
- Further Question.** There is now $\frac{8}{9}$ of $\frac{64}{81}$ left, i.e. $\frac{8}{9} \times \frac{64}{81} = \frac{512}{729}$ left. You can also think of this as $\left(\frac{8}{9}\right)^3$.

The BIG Answers:

Firstly, notice that at each step we are left with $\frac{8}{9}$ of what we started with.

- i. Doing it once more, we're left with $\frac{8}{9} \times \frac{512}{729} = \frac{4096}{6561} = \left(\frac{8}{9}\right)^4$.
- ii. Doing it 10 more times (from the initial 3 times), we're left with $\left(\frac{8}{9}\right)^{13}$.
- iii. Doing it 100 more times leaves $\left(\frac{8}{9}\right)^{103}$ left.
- iv. If we were to repeat this process forever, we would remove more and more of the individual carpet – however, after each step there is always some positive area of carpet left. Similarly to the Flummoxed Flea, for any tiny number you pick, the area that's left will eventually get smaller than this – so again here, we would say that the limit of $\left(\frac{8}{9}\right)^n$ is 0, or that there is no area left if we repeat the process an infinite number of times.

Take-home Problem

The factors of 2^{10} are 1, 2, 2^2 , 2^3 , ..., 2^{10} . So 2^{10} has 11 factors.

Week 5: Stand Up For Your Rights, Xs and Os, and the Number of Possible Paths.

Introduction

This is a Puzzle Station lesson. Students play two games (Stand Up For Your Rights, which anticipates binary, and can get messy – the answer also links to the Money Maximising Muddle; and Xs and Os), and do a combinatorics question (Number of Possible Paths).

Resources

- Calculators might be useful, especially for **Number of Possible Paths**.
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Stand Up For Your Rights.** Students have to work together to figure out the least number of moves it takes to go from all students sitting down, to the last student standing while all the rest sit, while obeying two rules. The answer is related to powers of 2, as was the Money Maximising Muddle. This activity can get a bit noisy!
- **Xs and Os.** This is Xs and Os, with a difference – as they're playing, students can swap between Xs and Os as they please. The winner is the one who places an X or an O on the board to make three in a row.
- **Number of Possible Paths.** This involves counting the number of possible paths from A to B in a given diagram. It foreshadows the Combinatorics lessons later.

Take-home Problem

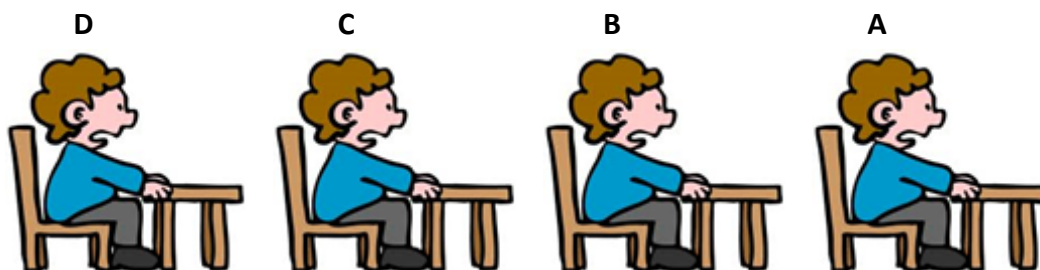
How many numbers from the set $\{1, 2, 3, \dots, 100\}$ are divisible by 3 or 4, but not both?



Stand Up For Your Rights!

This is a game for 1 or more people (though preferably more than 1!).

Everyone sits on a chair, one behind the other, all facing in the same direction:



(Neither the tables, nor the facial expressions, are mandatory.)

Aim: Get the last person in the row (here, person D) standing, and everyone else sitting, in as few moves as possible.

Rules:

- The first person (here, person A) can change from sitting to standing, or standing to sitting, whenever they like.
- Anyone else can change from standing to sitting, or sitting to standing, only if:
 - The person in front of them is standing, and
 - Everyone *else* in front of them is sitting.

So, for instance, if C wants to go from sitting to standing, then B must be standing, and A must be sitting.

Question: What is the least number of moves for:

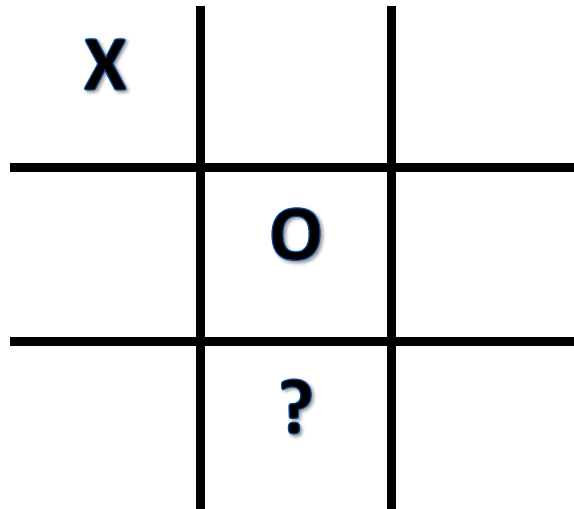
- One person?
- Two people?
- Three people?
- Four people?
- Ten people?

Can you see a pattern here? Do you recognise it?

Xs and Os

We're going to play Xs and Os, but with one big difference – at each turn, each player can put down either an X or an O.

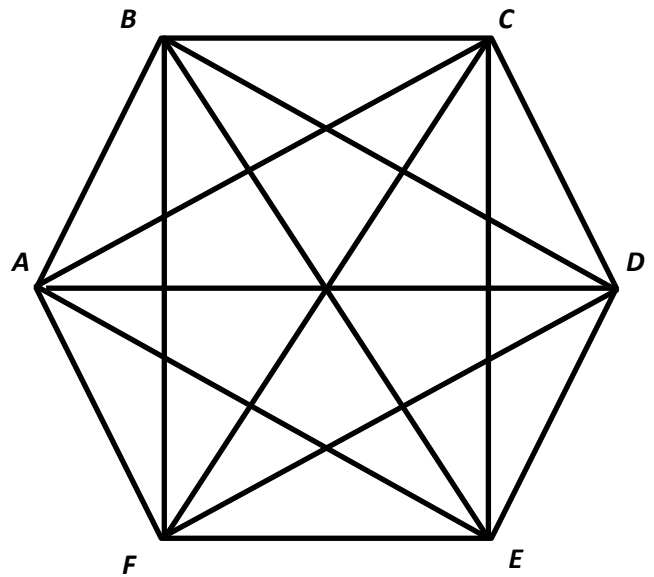
Whoever makes three in a row of either Xs or Os is the winner!



Number of Possible Paths

How many ways are there of getting from A to F , without passing through any of the vertices (i.e. corner points) more than once?

You can only change direction at a vertex (i.e. corner point). So, for instance, you can't head off from A towards D , and then, when you get to the centre, change direction and head towards C .



You might like to try it with a smaller shape, e.g. a square or pentagon, first. Can you see a pattern?

Extension Problem:

How many paths would be possible with a decagon (a 10-sided figure)?

Solutions

Puzzle Stations

Stand Up For Your Rights.

It should be fairly clear that for 1, 2, or 3 people, the minimum number of moves is 1, 3, or 7. Assuming these, and using the labelling from the question sheet, then we can tackle the problem with 4 people as follows:

Action	Number of moves
Get C standing (3 people involved – A, B, and C – so 7 moves), then get D to stand (1 move)	8
Get B standing (2 people involved – A and B – so 3 moves), then get C to sit down (1 move)	4
Get A standing (1 person, so 1 move), then get B to sit down (1 move)	2
Get A to sit down (1 move)	1

So the total number of moves for four people is $1 + 2 + 2^2 + 2^3 = 2^4 - 1$.

A similar argument will now work for 5 people, and 6 people, and so.

Note that we use the fact that $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$ in the argument – at each stage, we add in an extra step, to bring it up to the power of 2. This total popped up last week in the Money maximising Muddle.

So, for ten people we have $2^{10} - 1$ moves.

Xs and Os.

I haven't analysed this fully – it is a lot of fun to play, though. I'd be delighted if someone were to find a cast-iron winning strategy, or to show that no such strategy exists.

Number of Possible Paths.

We'll think of A as the *initial point*, F as the *end point*, and B, C, D, and E as the *middle points*.

It helps to look at how many different paths there are of each length. The length of a path is the number of edges it contains. So the path AF has length 1, while the path AEBF has length 3.

Note: The most vertices a path can contain is 6, so the longest a path can be is 5. (In order for a path to be length 6, it would have to end up back at A, rather than ending at F.)

Length of path	Number of such paths	Explanation
1	1	Just the path AF
2	4	From A, I can go to B, C, D, or E. Then I must go to F.
3	12	From A, I can go to B, C, D, or E: 4 choices. Next, I must go to another middle point. There are only 3 left. Finally, I must go to F. So $4 \times 3 = 12$.
4	24	From A, I have 4 choices (B, C, D, or E). Next, I have 3 choices left. Now I have 2 choices left. Finally, I must go to F. So $4 \times 3 \times 2 = 24$.
5	24	From A, I have 4 choices (B, C, D, or E). Next, I have 3 choices left. Now, I have 2 choices left. Now there is just 1 middle point left. Finally, I must go to F. So $4 \times 3 \times 2 \times 1 = 24$.

So there are $1 + 4 + 12 + 24 + 24 = 65$ different possible paths from A to F.

Extension Problem

For a ten-sided figure we can work the exact same way:

Length of Path	Number of Paths
1	1
2	8
3	$56 = 8 \times 7$
4	$336 = 8 \times 7 \times 6$
5	$1680 = 8 \times 7 \times 6 \times 5$
...	...

Continuing on in this way, we work out the number of paths for a ten-sided figure to be $1 + 8 + 56 + 336 + 1680 + 6720 + 20160 + 40320 + 40320 = 109601$.

Take-home Problem

There are 33 numbers divisible by 3 and 25 divisible by 4. However, there are 8 numbers divisible by 12, i.e. by both 3 and 4. So there are $33 - 8 = 25$ numbers divisible by 3 but not 4, and $25 - 8 = 17$ numbers divisible by 4 but not 3. So there are $25 + 17 = 42$ numbers divisible by 3 or 4 but not both.

Week 6: Beastly Brainteasers, Taxicab Challenge, and the Towers of Hanoi.

Introduction

This is a Puzzle Station lesson. Students complete some brainteasers, do some combinatorics by systematic listing, and try to solve the Towers of Hanoi (again involves powers of 2).

Resources

- Matchsticks would be useful for **Beastly Brainteasers**.
- Calculators for the **Beastly Brainteasers**.
- Box paper might be useful for the **Taxicab Challenge**.
- 1 or 2 sets of the **Towers of Hanoi** would be useful. If not available, they can be easily made by marking out three locations for the three pegs, and using things that get progressively smaller and can be laid on top of each other (e.g. a rubber, a notebook, a copybook, a textbook, etc) for the rings.
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Beastly Brainteasers**. Students try their hands at some brainteasers – these involve matchsticks, calculating areas, and a logic puzzle. They may need to be reminded that the area of a circle is πr^2 .
- **Taxicab Challenge**. This is a combinatorics question that we will return to – it asks how many ways can you get from A to B, in a city-like grid, if you only move up or to the right at each intersection.
- **Towers of Hanoi**. Students have to move a number of rings of increasing size from one peg to another, using a third peg. The catch is that a ring cannot be placed on top of a smaller one.

Take-home Problem

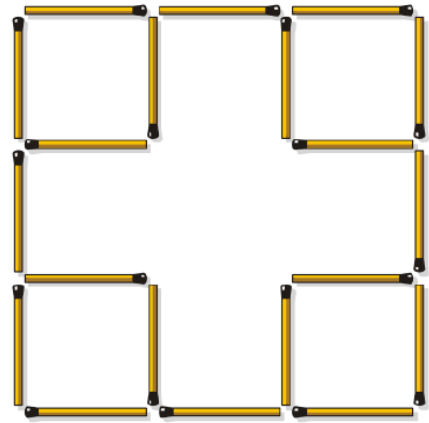
A rhombicosidodecahedron is a 3D shape, made up of 20 triangles, 30 squares, and 12 pentagons. How many edges does it have?

Beastly Brainteasers

Puzzle 1

On the right are twenty matchsticks, formed into 5 squares (one big and four small).

Can you move just two matchsticks so that you have seven squares?



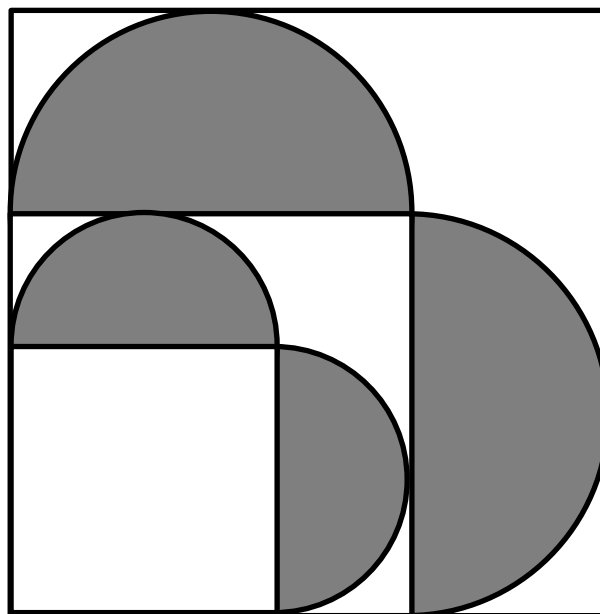
Puzzle 2

In front of you are a number of long fuses. You know they burn for exactly one hour each after you light them at one end. The entire fuse does not burn at the same speed though – for instance, it might take fifty minutes to burn half-way through one fuse, and only ten minutes to burn to the other end.

With your lighter and these fuses, how can you measure exactly one quarter of an hour, in order to cook your delicious oven pizza?

Puzzle 3

In the figure below, made up of squares and semicircles, which area is bigger – the grey area or the white area?




Taxicab Challenge

A taxicab metric is one where you can only move left, right, up, or down, just like in the map of Barcelona on the right.

John needs to get from Passeig de Gracia to his house. In order to go as fast as possible, he will only move East and North (i.e. right and up) – one such route is shown by the dotted line.



- (a) How many different possible routes does he have?
- (b) What if, instead of going over 2 and up 3 streets, he needed to go over 2 and up 4? How many different routes would be possible in this case?
- (c) How many routes would be possible if he needed to go over 2 and up 5?
- (d) Is there a pattern here? How many routes would be possible if he went over 2 and up 10?
- 

Extension Questions:

- (e)** What if he had to go over 3 streets and up 4? How many routes would be possible then?
- (f)** What if he had to go over 4 streets and up 6? How many routes would be possible then?

Towers of Hanoi

The Game.

You have a number of disks, all of different sizes, and three poles. Each disk can slot on top of any of the three poles. You need to move each disk from the first pole to the third pole. The rules are:

- You can only move one disk at a time.
- You can't place a larger disk on top of a smaller disk.



Challenge 1.

What's the least number of moves you need to complete the game if you have:

- 1 disk?
- 2 disks?
- 4 disks?
- 7 disks?

Challenge 2.

The puzzle was invented by Édouard Lucas in 1883, though he possibly based it on an old legend. In his version, there are three old posts and 64 golden disks in an Indian temple. The Brahmin priests are in the process of moving the disks from the first pole to the third. According to the legend, when the last move of the puzzle is completed, the world will end.

Assuming that each move takes 1 second (those are some fit Brahmin!), how many years will it take to finish the puzzle, once it's started?

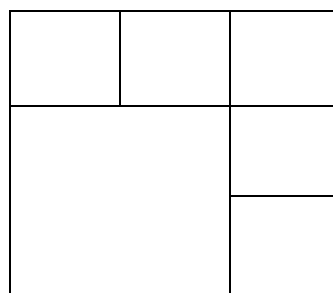
Challenge 3.

Suppose we introduced an extra pole, and kept the rules the same. Now what's the least number of moves you need to complete the game if you have:

- 1 disk?
- 2 disks?
- 4 disks?
- 7 disks?

Solutions

Puzzle Stations



Beastly Brainteasers.

1. Any rotation of the following will do:
There are five 1×1 squares, one 2×2 square and one 3×3 square.
2. Take 2 fuses, Fuse A and Fuse B. Set Fuse A burning from both ends simultaneously, and set Fuse B burning from one end at the same time. Fuse A will burn up completely after exactly half an hour. At this point, Fuse B will have exactly 30 minutes left to burn. At this point, light the unlit end of Fuse B. Now exactly 15 minutes will pass before Fuse B is entirely burnt up.
3. Let the side of the small square be 4 units – this means that all of the radii will be whole-number values. (So the radius of the circles are 2 and 3, and the sides of the squares are 4, 6, and 9.) Because we are interested in the fraction of the square that is shaded, we can use whatever side length we like.
It now turns out that the area of the whole shape is $9^2 = 81$ square units. The shaded region is composed of 2 circles, whose combined area is $\pi(3^2) + \pi(2^2) = 40.84$ to 2 decimal places, slightly over half the total area of the shape. So the shaded area is slightly greater.

Taxicab Challenge.

All of these can be done by systematic listing, and spotting patterns that arise. We will revisit this question in a few weeks when we look at nCr , so all of the answers here will also be given in terms of nCr .

- (a) We are being asked: in how many ways can the letters EENNN be arranged. By writing them out systematically, this can be seen to be $5C2 = 10$ ways.
- (b) In how many ways can EENNNN be arranged? $6C2 = 15$ ways.
- (c) In how many ways can EENNNNN be arranged? $7C2 = 21$ ways.
- (d) (a) to (b): up 5. (b) to (c): up 6. In a similar fashion, EENNNNNNNNNNN can be arranged in $21 + 7 + 8 + 9 + 10 + 11 = 66$ ways. (Also, $12C2 = 66$.)
- (e) EEEENNNN: $7C3 = 35$ ways.
- (f) EEEENNNNNN: $10C4 = 210$ ways. The method of systematic listing is pretty unwieldy here!

Towers of Hanoi.

Challenge 1.

The table outlines the least number of moves required for 1 – 7 disks.

Number of Disks	Least number of moves required	Least number of moves as a power of 2
1	1	$2^1 - 1$
2	3	$2^2 - 1$
3	7	$2^3 - 1$
4	15	$2^4 - 1$
7	127	$2^7 - 1$

Notice that, for n disks, we require $2^n - 1$ moves to complete the game.

The reason for this is that, if we can move, say, 5 disks from the 1st to the 3rd peg in 31 moves, then to move 6 disks we simply:

- Move the top 5 disks to the 2nd peg – this takes 31 moves.
- Move the bottom disk to the 3rd peg – 1 move.
- Move the top 5 disks from the 2nd peg to the 3rd peg – this take 31 moves.

Then the total number of moves is $31 + 1 + 31 = 63$. A more revealing way of adding these (requiring some indices) is as $(2^5 - 1) + 1 + (2^5 - 1) = 2(2^5) - 1$.

Adding the indices of the first two terms gives $2^6 - 1$, as we had hoped.

This trick works for any n , so as the $2^n - 1$ rule clearly holds for $n = 1$, an inductive argument can be used to prove the formula in general.

Challenge 2

For 64 disks, we require $2^{64} - 1$ moves. In years, this is $\frac{2^{64}-1}{60 \times 60 \times 24 \times 365}$, which is around 600 billion years!

Challenge 3.

The four-peg Towers of Hanoi is an *open problem* in mathematics – it has not yet been solved. That is, no formula has been found to tell what the least number of possible moves is for n disks. For 1, 2, 3, or 4 disks, the answers are 1, 3, 5, and 9, respectively.

Take-home Problem

Each edge of the 3D shape is made by joining 2 edges of 2D shapes together. So calculate the number of 2D edges, and divide by 2. This gives an answer of

$$\frac{(20 \times 3) + (30 \times 4) + (12 \times 5)}{2} = 120 \text{ edges.}$$

Week 7: Leaping Lizards, Cup Conundrums, and Safe Queens.

Introduction

This is a Puzzle Station lesson. Students play a game in Leaping Lizards, look at probability in Cup Conundrums, and try to solve the Safe Queens problem.

Resources

- Different coloured counters – white, green, blue, yellow, red – for **Leaping Lizards**. In case colour printing is not available, the board has letters on it for the different colours, so students could just use pieces of paper with the letters written on them as counters.
- Ten white and ten black counters or pieces of paper for **Cup Conundrums**.
- Chessboard and 8 pieces, to stand for the queens, in **Safe Queens**.
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Leaping Lizards**. This is a game where each student has a colour, and must try to get back to their own colour square by moving to the only free square. The main question is: is it always possible to finish the game.
- **Cup Conundrums**. In this puzzle students need to distribute black and white marbles between two cups in order to maximise or minimise the probabilities of certain events.
- **Safe Queens**. Students try to place n queens on an $n \times n$ chessboard (from 4×4 to 8×8), so that no queen is able to capture any other queen.

Take-home Problem

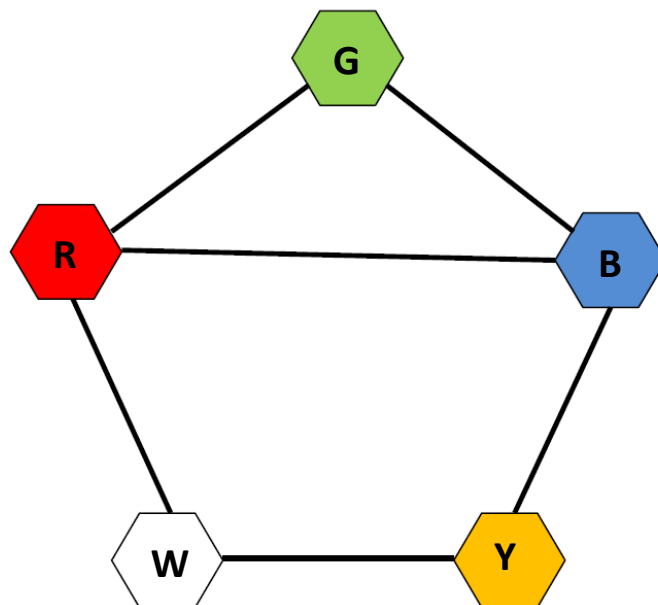
64 small cubes of side 1cm are joined together to make a bigger $4 \times 4 \times 4$ cube. The outside of this big cube is then painted. How many of the small cubes have exactly 2 sides painted?

Leaping Lizards

Each player gets a counter with a distinct colour at the start of this game – red, green, blue and yellow. No-one is white, so there are just 4 counters in play. At the start of the game, people start on random colours – no-one starts on their own colour. You can only move to an empty space, and along one of the edges.

The goal of the game is to place everybody in his/her own colour by allowing people to move along the paths.

Can you always finish the game, or does it depend on where players start?

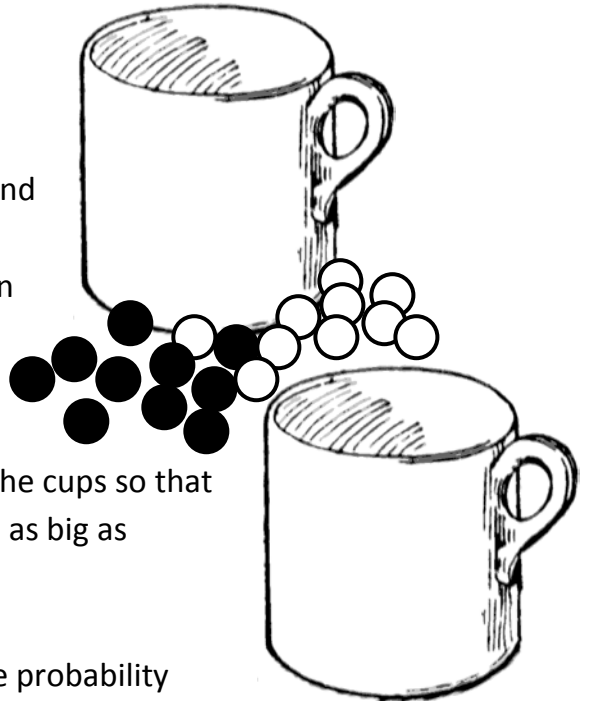


Extension Questions:

- a. In how many ways can the players place themselves on the board?
- b. If you add in an extra colour, say black, between white and yellow, can you finish the game then?

Cup Conundrums

Suppose I have 10 white marbles, 10 black marbles, and two cups. I am going to distribute the beads between the two cups. My friend Jane, who is blindfolded, then picks a cup at random, and from that cup, she picks a marble.



1. How should I distribute the marbles between the cups so that the probability that she picks a white marble is as big as possible? What would this probability be?
2. How should I distribute the marbles so that the probability that she picks a white marble is as small as possible? What would this probability be?

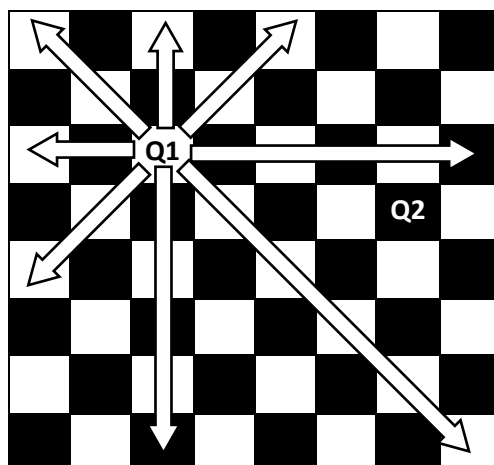
So, for instance, I could put all the white counters in the first cup, and all the black counters in the second. In this case, there's a $\frac{1}{2}$ chance that she picks the first cup (and a white counter), and a $\frac{1}{2}$ chance that she picks the second cup (and a black counter).

On the other hand, I could put 3 counters of each colour in the first cup, and 7 of each colour in the second. This time, there's a $\frac{1}{2}$ chance that she picks the first cup, and, if she does, a $\frac{1}{2}$ chance that she picks a white counter. This gives a $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ chance that she picks a white counter this way. Similarly, there's a $\frac{1}{4}$ chance that she picks the second cup, and then a white. So altogether, she has a $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ chance that she picks white, just like in the first example.

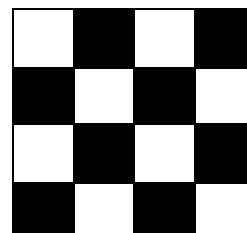
The question is: can you arrange the counters so that her chances of picking white go up, or go down, as much as possible?

Safe Queens

The queen is the most dangerous piece in chess – she can move along any row or column, and any diagonal. So, for instance, in the following diagram, Q1 (Queen1) is attacking all of the squares covered by the arrows. Because Q2 isn't on any of these squares, she is safe (and *vice versa*, so Q1 is safe from Q2).



1. Start with a 4×4 chessboard, as on the right. The aim is to place four queens on the chessboard, so that no queen can capture any other queen (hence the name of the game!).
2. Now try to do the same with 5 queens on a 5×5 chessboard – can you do it?
3. What about 6 queens on a 6×6 chessboard?
4. You guessed it – try 7 queens on a 8×8 chessboard!
5. And finally... try to place 8 queens on a full-sized, honest-to-goodness 8×8 chessboard, so that no queen can capture any other queen.



Extension Problem:

Okay, so it worked (hopefully!) for every board size from 4×4 to 8×8 . So, can you always place n queens safely on an $n \times n$ board, if n is at least 4? In other words, if we have the same number of queens as rows, can we always place them safely? Try it with 9×9 and 9 queens – is it possible to get a strategy that would work, no matter how big you make the chessboard?

Solutions

Puzzle Stations

Leaping Lizards.

For the sake of simplicity, we will only move counters in an anticlockwise motion around the outer pentagon. In order to win the game, we need the counters to be in the order YBGR (anticlockwise) – then we can easily shift them around so that they are in their proper places.

In order to solve the game, first of all write out the order the counters are in at the beginning, starting, say, from Y. So, for instance, you might have YGRB. The plan is to get from here to the required YBGR, by swapping the order of adjacent counters. So, for instance, if we swap R and B from our starting position we get YGBR. Now swapping G and B gives YBGR, as required.

In order to swap two adjacent counters, we rotate all counters until the two counters to be swapped are in the places marked B and G, while the place marked R is left free. The counter in the place marked B can now be moved to the place marked R to accomplish the swap. Using this method we can create YBGR from any starting position by performing successive swaps.

Extension Questions.

- (a) There are 5 options for the 1st counter, 4 for the 2nd, 3 for the 3rd, and 2 for the last, giving $5 \times 4 \times 3 \times 2 = 120$ different initial positions.
- (b) You can finish the game no matter how many extra places are added, as long as you always have a little triangle in which to perform your swaps.

Cup Conundrums.

1. Put 1 white counter into 1 of the cups, and the remaining 19 counters (9 white and 10 black) into the other. The probability of picking the cup with 1 counter in it is $\frac{1}{2}$, and if you pick this cup then you will definitely get a white counter. The probability of picking the other cup is similarly $\frac{1}{2}$, and if you pick this cup the chances of getting a white counter is $\frac{9}{19}$. So the probability of picking that cup and then picking white is $\frac{1}{2} \times \frac{9}{19} = \frac{9}{38}$. Thus the probability of picking white with this set-up is $\frac{1}{2} + \frac{9}{38} = \frac{28}{38} = 0.7368$, to 4 decimal places.
2. If you're allowed to leave 1 cup empty, then do that, and put all the counters in the other cup. Then in order to pick white you need to first pick the cup with counters in it, and then pick a white counter. The probability of this happening

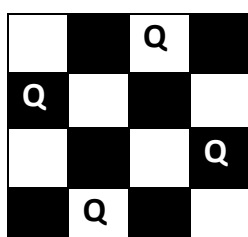
is $\frac{1}{2} \times \frac{10}{20} = \frac{1}{4} = 0.25$. If you're not allowed to leave one cup empty, then do the opposite of what you did in **1**. This will give a probability of $1 - 0.7368 = 0.2632$ of picking white, to 4 decimal places.

Safe Queens.

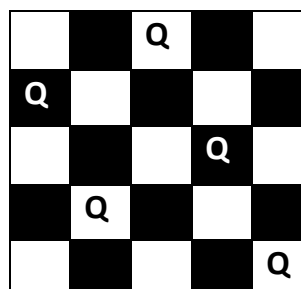
The positions for the queens are shown below. There are quite a number of distinct solutions for each size board. The ones below all share a similar staircase-style pattern, where the queens move in knight-like L shapes. Not all solutions have these features.

There's a nice Java applet at <http://www.math.utah.edu/~alfeld/queens/queens.htm> that shows, in real time, how one particular algorithm would search for solutions in an $n \times n$ board.

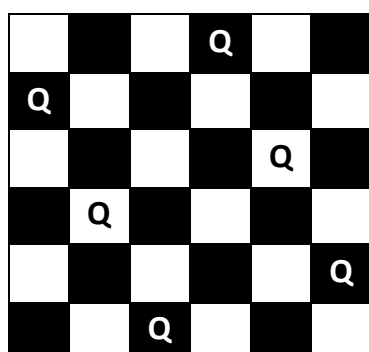
- 1.** For the 4×4 chessboard:



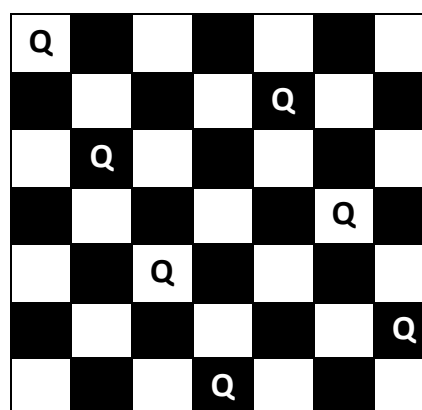
- 2.** For the 5×5 chessboard:



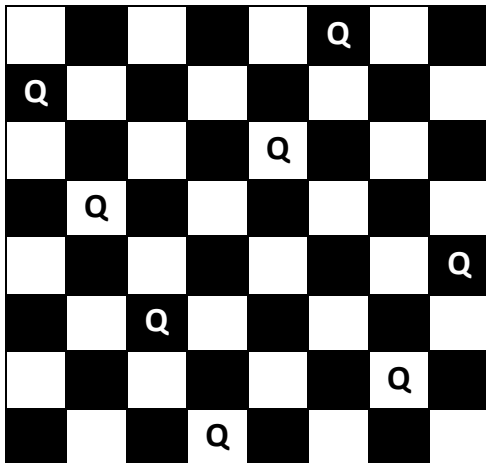
- 3.** For the 6×6 chessboard:



- 4.** For the 7×7 chessboard:



5. For the 8×8 chessboard:



Extension Problem. Using a similar, staircase-style approach, you can put n queens safely on an $n \times n$ board. It can be interesting to show exactly how to do it as n gets larger!

Take-home Problem

The only cubes with exactly 2 sides painted are the middle 2 cubes on an edge. As there are 12 edges to a cube, there are $12 \times 2 = 24$ cubes with exactly 2 faces painted.

Week 8: Crafty Card Tricks

Introduction

This is a Whole Class lesson. Students are taught three different card tricks – they are challenged to figure out how each one works. The first trick involves deception, while the other 2 are self-working tricks, i.e. the trick will work automatically, there is no deception or trickery involved. They are also given a card-related puzzle at the end of the lesson – it might be an idea to spend a minute or two showing them how most approaches to this problem won't yield a solution.

Resources

- A few packs of cards

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Whole-Class Lesson:

1. 3-Card Trick (10 mins)
2. 21-Card Trick (15 mins)
3. 9-Card Trick (25 mins)

Take-home Problem:

Blindfold Card Challenge

Whole-Class Lesson

1. 3-Card Trick (10 mins)

This is the only trick that requires deception – it's been put in here to make sure that students are paying attention! The trick involves putting cards in the middle of the pack, which then magically move to the top of the pack.

The teacher takes a deck of cards, and turns it face up. The trick is to take note of the card that will be second in the deck, once the deck is turned face down. Suppose this card is the 2 of Hearts.

The teacher says: "Let's do the trick with 2's. So, I'll go through the pack and pick out three 2's." The teacher then picks out the three 2's, other than the 2 of Hearts, which she leaves alone. She quickly shows the students the three 2's, then turns the pack face down, and puts the three 2's face down on the top of the pack.

Now, she taps the pack, to do the magic. Then she turns the top card face up – it's a 2 – and slides the next card face down in the middle of the pack. She turns the next card face up – another 2 – and slides the next card face down into the middle of the pack. Finally, she turns the next card face up – it's a 2 as well!

If students are paying attention, they will notice that the last 2 was not one of the original 2's that they were shown. If not, the trick could be done again, or students could be asked to try it for themselves. If they don't spot the trick, they won't be able to replicate it, though it can be interesting for them to try! Even if they do spot that the 3 cards turned up are not the original 3, they may not see how to do the trick until it has been shown a number of times.

2. 21-Card Trick (15 mins)

This is a very old trick – students may have seen it, but are generally unaware of why it works. One video (of many) demonstrating the trick can be seen at <http://www.youtube.com/watch?v=k7X6YtoMmNs>.

The teacher deals 21 cards face up onto the table in 3 columns of 7, dealing 1 card to each column at a time. (Deal to group 1, then group 2, then group 3. Then back to group 1, etc.) The teacher then turns her back and asks a student to point to a particular card. When the student has picked, and all the class have seen, the teacher turns back around. She asks which of the 3 piles the card is in. She then picks up the

three piles, making sure that the cards stay in order, and that the pile with the student's card is the middle one.

Next she repeats the process – she deals out the 21 cards in 3 equal piles, 1 card to each group at a time, and asks the student to indicate which pile contains his card. Again, she takes up the 3 piles, making sure the one with his card is in the middle.

Finally, the teacher performs the process for a third time. Now, when she puts the 3 piles together with the one containing the student's card in the middle, the student's card will be the 11th one of the 21, or the 4th one in the pile that contained his card. She can now perform the reveal any way she likes.

If students spot that the card is always the 4th one in the pile, they can try doing the trick themselves to see that it works. The challenge for them now is to explain how the trick works.

The trick here is that the pile with the student's card goes in the middle each time. So after the first go, the student's card is somewhere between the 8th and 14th cards, inclusive. Once the cards are dealt out for a second time, they will be arranged as below – the shaded cards are those that were in the column that was put in the middle after the first go.

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21

Now when the pile containing the student's card is put in the middle, the student's card is either the 10th, 11th, or 12th of the 21 cards. Finally, after the third go, the student's card must be the middle one of the pile that it's in, and the middle one of the 21 once this pile is put between the other two.

3. 9-Card Trick (25 mins)

There is a video of this trick at <http://www.youtube.com/watch?v=POQTbs2YfFs> – this is where I saw the trick for the first time.

The teacher gives a student 9 random cards, and have them pick one, say the 7 of Clubs. They then turn the pile of 9 cards face down, with the 7 of Clubs on top, and place the pile under the rest of the deck.

The teacher now deals out 4 piles from the top of the deck, turning up the cards as she goes. She counts down from 10 for each pile, and if the card she turns over matches the number she says, that is the last card in the pile, and she moves on to the next one. (Kings, Queens, and Jacks count as 10, and Aces as 1.) If she gets to 1 without meeting a match, she places a card face down on top of the pile, and moves on to the next pile. So, for instance, if the first 5 cards are, in order, 3-Hearts, 7-Clubs, 8-Hearts, Queen-Diamonds, Ace-Spades, then she deals out the 3 saying “ten”, the 7 saying “nine”, and the 8 saying “eight”. As these match, she leaves this pile, and starts a new pile. She deals out the Q saying “ten”, the A saying “nine”, and so on.

Now if there are cards facing up (i.e. cards that matched the number she was saying as she dealt out the 4 piles), she adds their values together. Suppose there are just 2 cards facing up – 2-Clubs and 6-Diamonds – and the other 2 piles each have a card face down on top of them. As $2 + 6 = 8$, the student’s card is the 8th card in what remains of the deck.²

Once again, students can try the trick for themselves to see that it works, but the challenge is to explain how it works. They should be encouraged to wonder if the 9 cards at the start is important – would it work if there were 10 cards dealt out at the start for them to pick from? Or 8 cards? Also, can you tell how many cards are in a bundle just by looking at the top number in the bundle? What if the card on top is turned over?

So, the trick: suppose the numbers on top of the 4 piles are a , b , c , and d . If a pile has a card face down on top of it, then we will say that the number on top is 0. (We

² One caveat: If (and this is highly unlikely) the teacher meets *no* card that matches the number she is saying at the time, then, instead of placing a card face down on the 4th pile, she says: “This is your card,” and turns it over to reveal that it is. However, the chances of this happening are very small (interesting question: what is the probability that this happens?), so it’s probably not worth mentioning until a solution has been gone through in full with the students.

presume initially that at least one of a , b , c , and d is not zero.) Then the first pile contains $11 - a$ cards, the second contains $11 - b$, and so on. (Check to see that you're happy with this!) So, between the 4 piles, we have dealt out:

$$11 - a + 11 - b + 11 - c + 11 - d = 44 - (a + b + c + d) \text{ cards.}$$

This means that there are $52 - [44 - (a + b + c + d)] = 8 + a + b + c + d$ cards left in the deck. Remember, the trick now says that the card chosen will be the $(a + b + c + d)$ th card. If we remove $a + b + c + d$ cards from the deck, we will have 8 left, and so our chosen card will be the last card removed, as predicted by the trick.

In the case of our caveat, when we come to the end of the 4th pile we have removed $11 + 11 + 11 + 10 = 43$ cards, leaving $52 - 43 = 9$ cards in the pack. Thus the next card is the chosen one.

Take-home Problem: Blindfold Card Challenge

A blindfolded man is handed a deck of 52 cards and told that exactly 10 of these cards are facing up. How can he divide the cards into two piles (possibly of different sizes) with each pile having the same number of cards facing up?

Hint: You don't just divide the deck into 2 piles – you must do something else as well. But remember, you can't look at the cards!

Clearly, dividing the pile in half won't help at all. Instead, he divides the pack into 2 piles, one of 10 cards and the other of 42. (It doesn't matter how he does this.) Now he turns the pile with 10 cards upside-down. The two piles will now have the same number of cards facing up.

To see why, suppose that once the pack is divided into the 2 piles, there are k cards facing up in the pile with 10 cards. This means that there must be $10 - k$ cards facing up in the other pile. Now, when he flips the pile with 10 cards upside-down, there will be $10 - k$ cards facing up in it, the same number facing up as are in the other pile.

Week 9: Elfen Fun, Christmas Tree Combinations, and Festive Brainteasers.

Introduction

This is a Puzzle Station lesson. Students solve some logic puzzles (Elfen Fun), do some work on combinatorics (Christmas Tree Combinations) and attempt some Festive Brainteasers.

Resources

- Calculators for Christmas Tree Combinations.
- Sheets of plain paper and a few scissors for the Festive Brainteasers (Q4).
- A copy of each of the Puzzle Station question sheets for each student.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Puzzle Stations (10-15 minutes each)

- **Elfen Fun.** Students solve a logic puzzle. There is a more challenging logic puzzle – a form of the Crocodile Dilemma – included in case they're finished early.
- **Christmas Tree Combinations.** Students solve questions based on combinatorics. One of the questions (Q3) also introduces Pascal's Triangle.
- **Festive Brainteasers.** Two of these are arithmetic problems (Q1 and 2), Q3 is a logic problem, and Q4 involves cutting a hole in an A4 sheet large enough for someone to walk through.

Take-home Problem

Mary has got 8 circular hula-hoops for Christmas, and has placed them on the ground so that each pair of hula-hoops overlaps. What is the greatest possible total number of intersection points of the eight hula-hoops?

Elfen Fun

You have landed at the North Pole, and come across five elves. They all look the same, but you have a feeling that some of them are good elves, and always tell the truth, while some of them are bad elves, and always lie. Unfortunately, there's no way to tell them apart just by looking at them – they all look cute and helpful!



- The first elf says: *I am a good elf!*
- The second elf says: *At least three of us are good elves!*
- The third elf says: *Careful! The first elf is a bad elf!*
- The fourth elf says: *At least three of us are bad elves!*
- The fifth elf says: *We are all bad elves!*

So:

1. How many elves are good, and how many elves are bad?
2. For some of the elves, it is impossible to tell if they are good or bad. Which ones are these, and why can't you tell?

Extra Question:



The Grinch, who hates Christmas, has kidnapped one of the good elves. Santa begs the Grinch to return the elf. The Grinch, whose heart was two sizes too small, says the following:

"If you guess correctly what I will do with your Elf, then I will return him. However, if you don't predict his fate correctly, I shall eat him!"

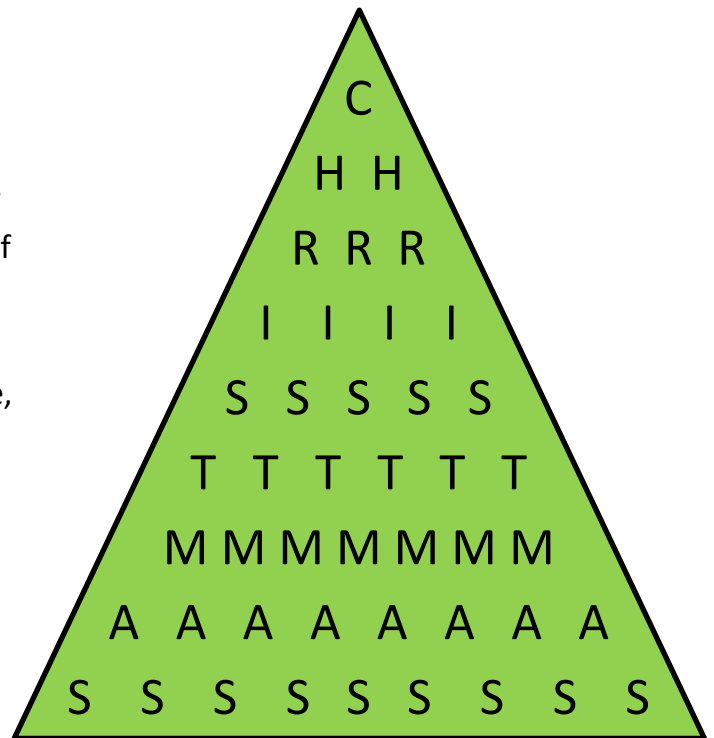
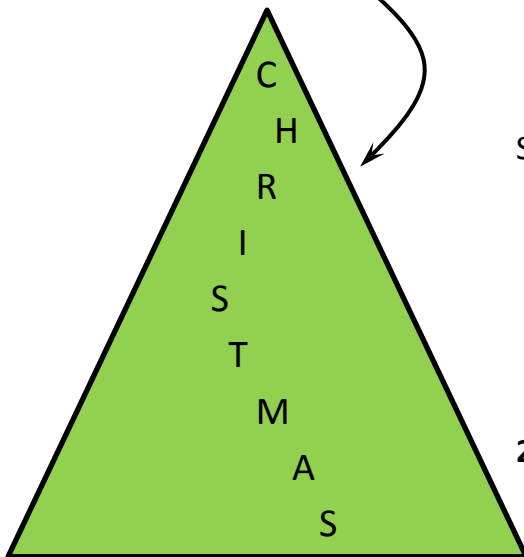
What should Santa say to the Grinch in order to save the Elf?

Christmas Tree Combinations

The aim of this puzzle is to maximise the number of presents John will get.

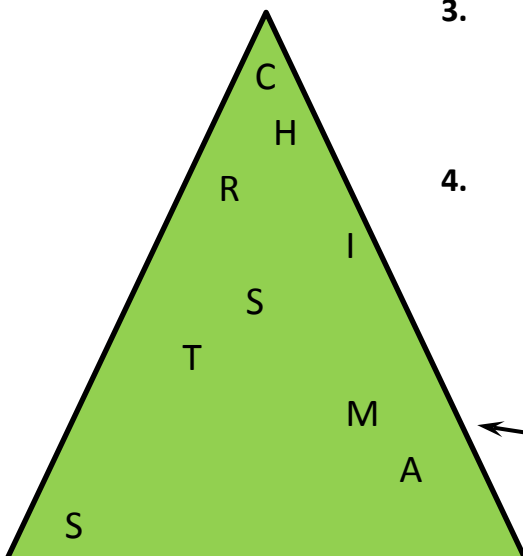
John's mother tells him that the number of presents he gets will be the number of ways you can spell the word 'Christmas' using the image on the right, by starting at C and moving down one row at a time, diagonally to the left or to the right.

For example:



So:

1. What's the maximum number of presents John can get?
Hint: Instead of doing it straight away with 9 rows, try doing it with just two rows first, then three rows, then four, and so on. Is there a pattern?
2. Replace each letter by the number of routes from the top to that letter. Do you notice any patterns here?
3. Now colour in all of the odd numbers in your answer to 2 above. Does this remind you of any shape we've seen before?
4. Finally, how many presents could he get if he didn't have to move just one step left or right, but if he could choose *any* letter from each row, as on the left?



Festive Brainteasers

1. Unfortunately the elves have lost their tape measure. It's unfortunate because they need to measure the size of the box for this year's top present, the Monster Madness Mobius Mobile. Luckily, they have remembered that the top has an area of 720cm^2 , the side has an area of 800cm^2 , and the end has an area of 360cm^2 . What are the dimensions of the box?



2. In order to calculate the amount of reindeer feed needed for Christmas Eve, Santa has to do some pretty complicated sums. He has scribbled one part of the sum on an envelope as follows:

$$\begin{array}{r} 2,147 \\ \times 3,725 \\ \hline 22,084,429 \end{array}$$

Unfortunately, while the lines have the correct digits, the first two lines have their digits in the wrong order. Nonetheless, the answer is correct. Can you work out what two numbers Santa actually multiplied together to get 22,084,429 as an answer?

3. Mrs Claus always sneezes just before it starts snowing. She just sneezed.

"This means that it's going to start snowing", thinks Santa. Is he correct?

4. Using a pair of scissors, cut a hole in a regular A4 piece of paper through which Santa could pass. (Assume that Santa is pretty big – I'd say about 6'2", and about 18 stone.)



Solutions

Puzzle Stations

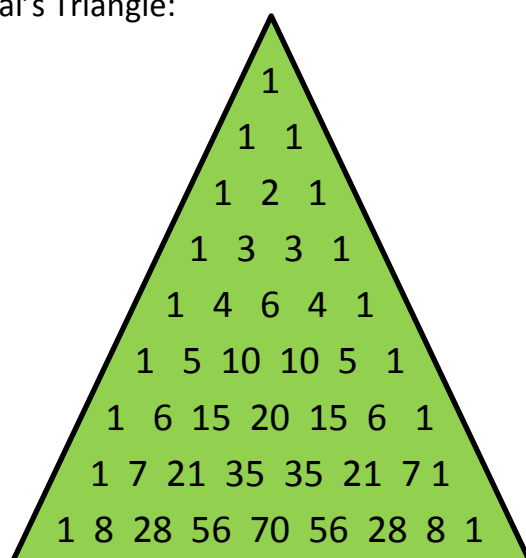
Elfen Fun.

1. The 2nd and 4th elves can't both be telling the truth. On the other hand, they can't both be lying either. So one is telling the truth, and the other is lying. This means that the 5th elf must be lying, when he says that they are all bad elves. Similarly, of the 1st and 3rd elves, one must be telling the truth and one must be lying – they can't both be telling the truth, and neither can they both be lying. This means that 3 elves are good, and 2 are bad.
2. This means that the 2nd elf is telling the truth, and the 4th elf is lying. However, we still have no idea which one is the good elf and which the bad between the 1st and 3rd elves.

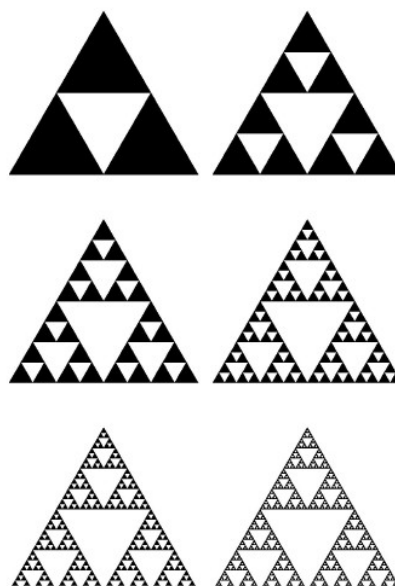
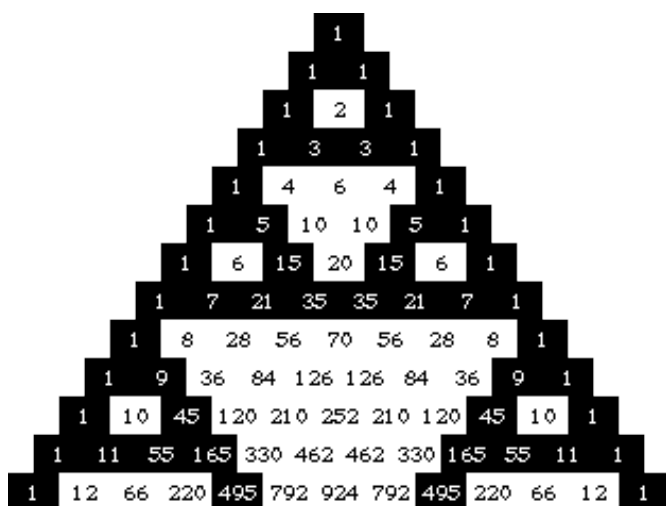
Extra Question: This is an old paradox (Google the crocodile paradox) – general agreement is that there is nothing Santa can say to save the Elf, though it's fun to try out different possibilities!

Christmas Tree Combinations.

1. As he moves from each row to the next, he can either move to the left or the right. This means that the maximum number of presents he can get is $2^8 = 256$.
2. This gives Pascal's Triangle:



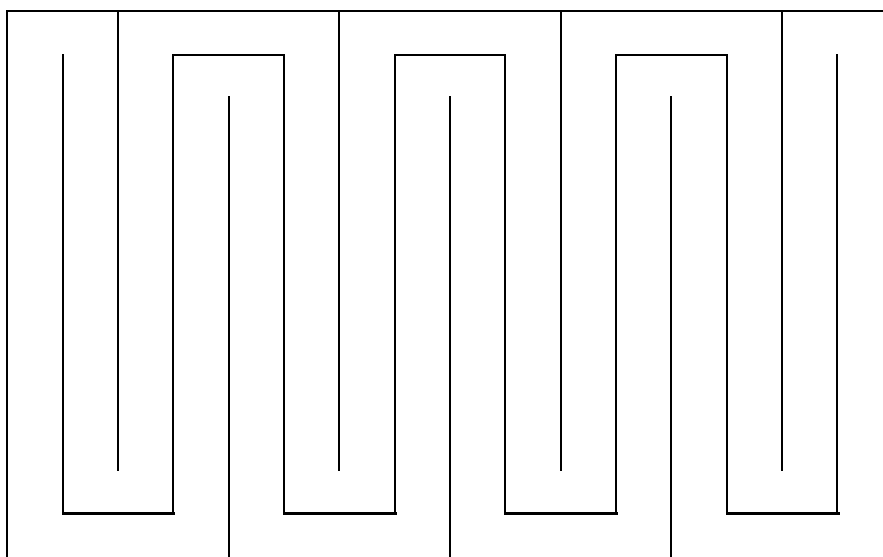
3. Colouring it gives Sierpinski's Triangle – this is the triangle version of the square we met in Lesson 4. On the next page is a version of the first 13 rows of Pascal's Triangle with the odd numbers coloured black, as well as a picture of the first 6 iterations of Sierpinski's Triangle.



4. In this case, he would have 2 choices at the H row, 3 at the R row, and so on. So he would have $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362880$ presents.

Festive Brainteasers.

1. The height multiplied by the side of the base is 360 cm^2 , and their ratio is $\frac{800}{720} = \frac{10}{9}$. So if the height is $10x$ then the side of the base is $9x$. The area they produce is then $(10x) \times (9x) = 90x^2 = 360$, so $x = 2$. This means that the height is 20 cm and the side of the base is 18 cm, so the other side of the base must have a length of 40 cm.
2. Because the last digit of the answer is a 9, the last digit of each of the smaller numbers must be a 7. Now, simply try dividing the 6 permutations of the first number that end in 7 into the answer, to see which one gives the correct answer. (The 6 permutations are 2147, 2417, 4127, 4217, 1247, and 1427.) It turns out that $22084429 \div 4217 = 5237$, giving the answers.
3. This is not a logical conclusion to draw. Mrs Claus might also start sneezing when there's pepper around, or when she smells tomatoes, or if it's a Tuesday. Just because she sneezed, it doesn't mean it's because it's snowing – it could be for one of the other reasons.
4. This can be done in a number of ways – the important thing is not to end up with just a big long strip of paper. One such solutions is top cut along the following lines (the pattern can be continued to give as long a loop of paper as is necessary):



Take-home Problem

The maximum number of points of intersection is 56. If we lay down a single circle, there are 0 points of intersection. Another circle lying on top will give 2 points of intersection. A further circle will intersect each existing circle at most twice, giving 4 extra points of intersection, for a total of $2 + 4 = 6$. A 4th circle will intersect each of the existing circles twice, giving a total of $2 + 4 + 6 = 12$. This pattern continues, so that with 8 circles we have a maximum of $2 + 4 + 6 + 8 + 10 + 12 + 14 = 56$ distinct points of intersection.

Week 10: Commencing Combinatorics: nCr

Introduction

This is a Whole-Class lesson. Students have already met the basic idea in Combinatorics (footnote 1 on next page) in some earlier lessons. Here we look in detail at this idea, bring in the idea of combinations versus permutations (i.e. order doesn't matter versus order does matter), and, at the end of the lesson, introduce the nCr notation.

Resources

- A calculator for each student with an nCr button, for the end of the lesson.
- A copy of the Commencing Combinatorics Question Sheet, one per student.
- A few packs of cards.

Activities

Warm-Up Problem (5-10 minutes): Last week's take-home problem.

Whole-Class Lesson:

1. Introduction – Fundamental Principle of Counting (5 mins)
2. Permuting 2 objects (10 mins)
3. Combining 2 objects (5 mins)
4. Permuting 3 objects (10 mins)
5. Combining 3 objects (10 mins)
6. Combining in General (10 mins)
7. nCr and Recap (5 mins)

Take-home Problem:

Finish worksheet.

Whole-Class Lesson

1. Introduction – Fundamental Principle of Counting³ (5 mins):

(a) Recap puzzles for which they have used this idea:

- i. Week 1: Magic Squares: finding the total number of ways of filling in 3×3 squares with the numbers 1 – 9 (though most of them are not magic!)
- ii. Week 5: Number of Possible Paths
- iii. Week 9: Christmas Tree Combinations

2. Permuting 2 objects (10 mins):

(a) Give each student 4 cards – how many ways can we put down 2 of them, in order? So, e.g., Ace Queen is different to Queen Ace. Get student to list them, as systematically as possible! So we'd be looking for something which would suggest the answer of 4×3 , like:

AK	KA	QA	JA
AQ	KQ	QK	JK
AJ	KJ	QJ	JQ

(b) Give each student an extra card – now how many ways can we put down 2 from the 5 that we have? Do a whole new list, don't just add on to the old one.

(c) How about with 6 cards? 7 cards?

(d) What about from a whole deck of 52 cards?

3. Combining 2 objects (5 mins):

(a) Back to 4 cards. How many ways can you deal a pair, if you don't care about order? Why is this different? How is it related to answer from Activity 2? Look at the list you wrote first time, and see how you'd need to change it to answer this new question. It might be useful this time to organise the list like the table below. There are no pairs on the diagonal, because a card can't be in a pair with itself. Each pair above the diagonal is now considered the same as its

³ The Fundamental Principle of Counting is the idea that:

- If I can do the first thing in a different ways, and
- The second thing in b different ways, then
- I can do the first thing, followed by the second thing, in $a \times b$ ways.

image, if we were to fold the table along the diagonal. So we only have to count the pairs above the diagonal. This is $3 + 2 + 1 = 6$, or half of the answer from Activity 1, i.e. $\frac{4 \times 3}{2} = 6$.

	A	K	Q	J
A		AK	AQ	AJ
K	KA		KQ	KJ
Q	QA	QK		QJ
J	JA	JK	JQ	

(b) What about dealing a pair from 5 cards? We need to add a column and a row to our table, and again it should be clear that there are $4 + 3 + 2 + 1 = 10$, or $\frac{5 \times 4}{2} = 10$, distinct pairs.

(c) Now what about 6 cards? 13 cards? 52 cards?

4. Permuting 3 objects (10 mins):

(a) Back to 4 cards. In how many ways can I put down 3 in order? If we list them, as systematically as possible, we should see that each triple occurs 6 times. Can students explain this 6, in terms of what we've met already? (It is $3 \times 2 \times 1$ – permuting 3 cards.)

(b) What about with 5 cards? 6 cards? One deck? Two decks (where we can tell cards from the 2 decks apart, so there are 104 distinct cards)?

5. Combining 3 objects (10 mins):

(a) In how many ways could you deal hands of 3 from 4 cards, where you don't care about order? Look back at the related list you made in Activity 4 – how would you have to change this list to answer this new question?

(b) In how many ways could you deal hands of 3 from 5 cards? List out all the different possible hands, and check by calculation, using the answer from Activity 4.

(c) How about dealing hands of 3 from 6 cards? 13 cards? A full deck?

(d) So: can we write down a general law for dealing hands of \blacksquare cards from a stack of \square cards?

6. Combining in General (10 mins):

- (a) Students to work through worksheet. If they're finished very quickly, they could devise their own questions.
- (b) Students could get up and do the handshaking question – question (f) – themselves.

7. nCr and Recap (5 mins):

- (a) Go back to early question: if you have 5 cards, how many ways can you deal hands of 3 cards? This can be done on the calculator using the nCr button. Get students to check this, and to check the answers to the questions they have already worked out, to make sure they get the same answers.

COMMENCING COMBINATORICS

Some Card Conundrums:

- (a) How many ways are there of dealing a hand of 5 cards for Poker from a deck of 52?
- (b) How many ways are there of dealing a hand of 7 cards for Rummy from a deck of 52?
- (c) Ahmed, Bob, Carla and Diamante are playing Cheat with a small pack of 20 cards, so they'll need 5 cards each.
 - i. In how many ways can Ahmed be dealt his hand of 5 cards, from 20?
 - ii. Given that Ahmed has been dealt his hand of 5, in how many ways can Bob be dealt his 5 cards? (How many cards are left in the pack?)
 - iii. Given that A and B have been dealt their hands, in how many ways can Carla be dealt her 5 cards? (How many are left in the pack now?)
 - iv. Given that A, B, and C have been dealt their hands, in how many ways can Diamante be dealt her 5 cards? (Now how cards many are left?)
 - v. So in how many different ways can the 20 cards be dealt out in hands of 5 to A, B, C, and D?
 - vi. How many ways would there be if they used a full pack of 52 cards?
- (d) Xerxes, Yolanda, and Zena are playing Poker, so they'll need 5 cards each.
 - i. How many different hands can Xerxes be dealt?
 - ii. How many different hands can be dealt to the players?
- (e) John has 10 cards.
 - i. How many different hands of 2 cards can be dealt?
 - ii. How many different hands of 8 cards can be dealt?
 - iii. Is there a connection between these two answers? Can you explain it?
 - iv. What other number of cards in a hand would be connected in the same way as 2 and 8?

More Combining Challenges:

- (f) There are 10 people in a room.
 - i. If each person shakes hands with everyone else exactly once, how many handshakes take place?
 - ii. Can you explain why this answer is also given by $1 + 2 + 3 + \dots + 9$?
 - iii. Does this remind you of anything you've met in these circles?
- (g) There are 20 people in a squad.
 - i. In how many ways can a team of 11 be chosen?
 - ii. In how many ways can the team be chosen if the captain has to be included in the team? (Hint: ask yourself how many spaces are left on the team, and how many people are left to fill the spaces.)
- (h) There are 42 numbers in a drum – 7 will be picked out for the Lotto.
 - i. How many different combinations are possible?
 - ii. Once the 7 numbers are chosen, a bonus number is then picked from the numbers remaining in the drum. How many different combinations of the 7 winning numbers plus bonus number are possible?



Solutions

1. Introduction – Fundamental Principle of Counting:

- i. **Magic Squares:** There are 9 choices for the first square, 8 for the second, 7 for the third, ..., and 1 for the ninth square. So there are $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$ different possible magic squares.
- ii. **Number of Possible Paths:** Calculate the number of paths of each length separately. So, for instance, the number of paths that go through six edges is $4 \times 3 \times 2 \times 1 \times 1 = 24$ different possible paths.
- iii. **Christmas Tree Combinations:** (i) At each stage, you have 2 choices – move down to the left, or down to the right. So there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$ different possible paths. (iv) Now, you have 1 choice for the C, 2 for the H, 3 for the R, and so on. So there are $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362,880$ different possible paths.

2. Permuting 2 objects

- (a) 4 cards: This can be done in $4 \times 3 = 12$ different ways. If the cards are A, Q, K, J, then the list is: {AQ, AK, AJ, QA, QK, QJ, KA, KQ, KJ, JA, JQ, JK}.
- (b) 5 cards: This can be done in $5 \times 4 = 20$ different ways. If the cards are A, Q, K, J, 10, then the list is: {AQ, AK, AJ, A10, QA, QK, QJ, Q10, KA, KQ, KJ, K10, JA, JQ, JK, J10, 10A, 10Q, 10K, 10J}.
- (c) 6 cards: $6 \times 5 = 30$ different ways.
7 cards: $7 \times 6 = 42$ different ways.
- (d) 52 cards: $52 \times 51 = 2,652$ different ways.

3. Combining 2 objects

- (a) 4 cards: Each pair appears twice in the list in Activity 2 – e.g. AQ and QA, or KJ and JK. Because we count AQ and QA as the same pair of cards, we have twice as many answers as we need. So the correct answer here is: $\frac{4 \times 3}{2} = 6$ different pairs can be dealt.

You could also approach this question by saying:

- a. The A can appear in 2 pairs: {AQ, AK, AJ}
- b. Removing the A, the Q can appear in 2 pairs: {QK, QJ}
- c. Removing the Q, the K can appear in 1 pair: {KJ}
- d. Removing the K leaves the J on its own, so no more pairs can be made!

This means that the answer can also be written as $1 + 2 + 3 = 6$. It might be worth reminding students that $1 + 2 + 3 = \frac{3 \times 4}{2}$, and that this holds in general.

If the section on arithmetic series has been done, it might be worth recalling that briefly here, and showing that both ways of thinking about the problem yield the same answer.

- (b) 5 cards: Again, each pair is counted twice. So the answer here is $\frac{5 \times 4}{2} = 10$ different possible pairs.
- (c) 6 cards: $\frac{6 \times 5}{2} = 15$ different possible pairs.
- 13 cards: $\frac{13 \times 12}{2} = 78$ different possible pairs.
- 52 cards: $\frac{52 \times 51}{2} = 1,326$ different possible pairs.

4. Permuting 3 objects

- (a) 4 cards: There are $4 \times 3 \times 2 = 24$ different ways to do this. If the cards are A, Q, K, J, then there are 6 ways with any given first card, e.g. with A first we have: {AQK, AQJ, AKQ, AKJ, AJQ, AJK}.
- (b) 5 cards: $5 \times 4 \times 3 = 60$ different ways.
- 6 cards: $6 \times 5 \times 4 = 120$ different ways.
- 1 deck: $52 \times 51 \times 50 = 132,600$ different ways.
- 2 decks: $104 \times 103 \times 102 = 1,092,624$ different ways. (This presumes that we can tell the difference between, say, the 3 of clubs in deck A and the 3 of clubs in deck B. If we can't, then the question becomes a lot trickier – it might be worth discussing this.)

5. Combining 3 objects

- (a) 4 cards: Each hand of 3 cards – e.g. A, Q, K – appears 6 times in the list we made in Activity 4. So the answer is $\frac{4 \times 3 \times 2}{6} = 4$ different hands.
- The thing to spot here is that the 6 comes from the fact that, when we've been dealt a hand of 3 cards – e.g. A, Q, K – there are $3 \times 2 \times 1 = 6$ ways that we can put them in order. This means that each hand appears 6 times in the list in Activity 4, so we must divide by 6 to take out these repetitions.
- (b) 5 cards: Again, each hand of 3 would appear $3 \times 2 \times 1$ times if we were to list them in the previous activity, so the answer is $\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$ different hands.
- (c) 6 cards: $\frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$ different hands.
- (d) 13 cards: $\frac{13 \times 12 \times 11}{3 \times 2 \times 1} = 286$ different hands.
- (e) 52 cards: $\frac{52 \times 51 \times 50}{3 \times 2 \times 1} = 22,100$ different hands.
- (f) General form: the number of ways we can deal hands of \blacksquare cards from a stack of \square cards will be a fraction.

- The bottom line will be all the numbers up to ■ multiplied together – this removes repetitions
- The top line will have ■ numbers in it, starting at □ and working down. So, e.g., dealing hands of 5 (■) cards from a pack of 21 (□) can be done in $\frac{21 \times 20 \times 19 \times 18 \times 17}{5 \times 4 \times 3 \times 2 \times 1} = 20,349$ different ways.

6. Combining in General – WORKSHEET

(a) $\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$ different hands.

(b) $\frac{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times 46}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 133,784,560$ different hands.

(c) i. $\frac{20 \times 19 \times 18 \times 17 \times 16}{5 \times 4 \times 3 \times 2 \times 1} = 15,504$ different hands.

ii. There are 15 cards left in the pack, so $\frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1} = 3,003$ different hands.

iii. There are now 10 cards left, so $\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$ different hands.

iv. There are now 5 cards left, so there's only 1 way that D can be dealt these 5 cards! Or, if you like, $\frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 1$ different way.

v. They can be dealt their hands in $15,504 \times 3,003 \times 252 \times 1 = 11732745024$ different ways. (You might need the Scientific Calculator on the computer for this!)

vi. It might be worth coming back to this, once you've met the nCr notation – in that form, the answer is:

$${}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} = 5.36 \times 10^{28}$$

or, roughly, 50,000, ...,000 different ways, where there are 28 zeros!

(d) i. X: $\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$ different hands.

ii. Y: 47 cards left, so $\frac{47 \times 46 \times 45 \times 44 \times 43}{5 \times 4 \times 3 \times 2 \times 1} = 1,533,939$ different hands.

Z: 42 cards left, so $\frac{42 \times 41 \times 40 \times 39 \times 38}{5 \times 4 \times 3 \times 2 \times 1} = 850,668$ different hands.

So total is $2,598,960 \times 1,533,939 \times 850,668 = 3.39 \times 10^{18}$ different ways to deal the 3 hands, or, roughly, 3,000, ...,000 different ways, where there are 18 zeros.

(e) i. $\frac{10 \times 9}{2 \times 1} = 45$ different hands.

ii. $\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 45$ different hands.

iii. They're the same! Choosing the 2 cards we will take is really the same as choosing the 8 cards we leave behind.

iv. 1 and 9, 3 and 7, 4 and 6. And, interestingly, 0 and 10. We can work out how many ways we can deal hands of 10 cards – can we do the same

with hands of 0 cards? Do we run into trouble with our method here?
(This is part of the reason why $0!$ is defined to be 1.)

- (f) i. This is the same as 10 cards, how many pairs can be dealt. So:

$$\frac{10 \times 9}{2 \times 1} = 45 \text{ handshakes.}$$
- ii. First person shakes hands with 9 people, next person then goes and shakes the remaining 8 people's hands, and so on.
- iii. Arithmetic sequences. Or adding the numbers from 1 to 100.
- (g) i.
$$\frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 167,960 \text{ different ways.}$$
- ii. There are 10 spaces left on the team, and 19 people left to choose from. So:
$$\frac{19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10}{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 92,378 \text{ different ways.}$$
- (h) i.
$$\frac{42 \times 41 \times 40 \times 39 \times 38 \times 37 \times 36}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 26,978,328 \text{ different ways.}$$
- ii. There are 35 numbers left in the drum, so once the 7 numbers are picked, there are 35 choices for the bonus number. So altogether there are $26,978,328 \times 35 = 944,241,480$ different possible outcomes.

Week 11: Continuing Combinatorics – “Sober or Blotto, Remember the Lotto”

Introduction

This is a Whole-Class lesson. It continues on from the previous lesson, and looks at how combinatorics can be applied in many different situations. Two different types of application are looked at: using nCr to count the number of different possible sequences, e.g. of black and white coins, or DNA (Sections 2 and 3); and using nCr to pick from different groups, e.g. picking 4 boys and 5 girls, or picking 4 winning numbers and the bonus number (Sections 4 and 5).

Resources

- Students should have their calculators.
- Some counters of different colours might be handy for Section 2.
- A copy of the Question Sheet (3 pages), one per student.

Activities

Whole-Class Lesson:

1. **Recap** (5 mins)
2. **Black and White** (10 mins)
3. **Taxicab Challenge** (10 mins)
4. **Picking From Groups** (10 mins)
5. **Sober or Blotto, Remember the Lotto** (15 mins)

Take-home Problem:

Finish worksheet.

Whole-Class Lesson

1. Recap (5 mins):

- (a)** Recap on idea of nCr – picking two cards, three cards, etc from 52.
- (b)** Simple Lottery problems:
 - (i)** If I am picking 3 numbers from 25, how many different outcomes are possible?
 - (ii)** Which of these would make it more difficult to win: Have people pick 4 numbers from 25, or have them pick 3 numbers from a total of 30?
Guess, then check!
 - (iii)** If the tickets cost €1 each, what would be a fair prize for the jackpot in the original question?

2. Black and White (10 mins):

- (a)** Suppose we want to line up some black and white counters in order – in how many ways can this be done if:
- (i)** There are 3 white and 2 black counters?
 - (ii)** There are 5 white and 8 black counters?
 - (iii)** There are 12 white and 17 black counters?

Students should try (i) by listing all outcomes, but what we're looking for is a method that will use nCr , and will do (iii) for us easily. So, after doing (i), students should ask themselves: what was common to all of the possible outcomes? How might this be used to generate a method to solve the other questions?

- (b)** This kind of method can be used in a number of different areas – try Questions 1 to 4 on the Questions Sheet.

3. Taxicab Challenge (10 mins):

- (a)** Recall the Taxicab challenge question: how many ways are there for a car to get from the bottom dot to the top one, if it can only drive up and to the right? One such path is indicated.
- Ask students to quickly try the question again. Then ask: would their method work if the cab needed to go over 10 streets and up 14? Again, we are looking for a method using nCr .

Hint: using the letters U (Up) and R (Right), write out the above path. (It's RUURU.) Now, can we use the method we just met?



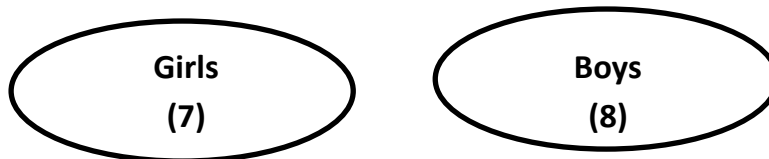
- (b) Now try Question 5 on the Question Sheet. Note that Q5(v) is extremely tricky – this might be left as a take-home challenge for the very good students.

4. Picking From Groups (10 mins):

- (a) Quick recap – suppose I have 4 choices for my meal (burger, nuggets, wrap, salad), and 3 choices for my drink (coke, orange, 7up). How many different ways can I choose the two together? Do I add or multiply? Why?
- (b) Suppose we have a group of 7 girls and a group of 8 boys. In how many ways can we pick a team of 5 from these, if:
- (i) We need 2 girls and 3 boys?
 - (ii) We need 3 girls and 2 boys?
 - (iii) We need just 1 girl?

Apart from using nCr twice here, the tricky thing is to see what to do with the two answers we get – we do exactly what we did with the 4 meals and 3 drinks in (a).

Also, it can be helpful here to draw two circles for the two groups:



How many do we want to choose from the first group? And how many are there in there to choose from? Now do the same with the second group, and combine the answers as in (a).

- (c) Now try Questions 6 and 7 on the Question Sheet.

5. Sober or Blotto, Remember the Lotto (15 mins):

- (a) Try Question 8 – parts (i) to (vi) are very similar to the previous section. The key insight is that, instead of Boys and Girls, we should split the numbers into Winning and Non-Winning. You might let students discuss it for a few minutes before suggesting this. Other than that, it works in the same way.
- It might be worth writing down a particular draw (6 WNs and 1 BN), and then writing out some examples that match, say, 4 WNs and the BN, just so they see that these can be done in multiple ways, and that it's the freedom to select different losing numbers that really drives up the number of ways.
- (b) Now try Question 9 – here, we need to split the Main Draw into Winning and Non-Winning, and split the Lucky Star Draw into Winning and Non-Winning, and then put everything together at the end. Again, let students discuss this before telling them.

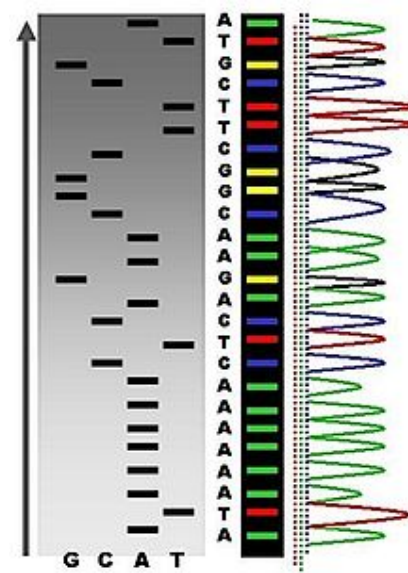
CONTINUING COMBINATORICS:

Some Easy-Peasey Preliminaries

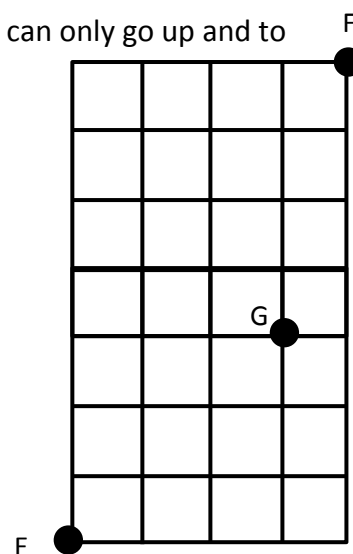
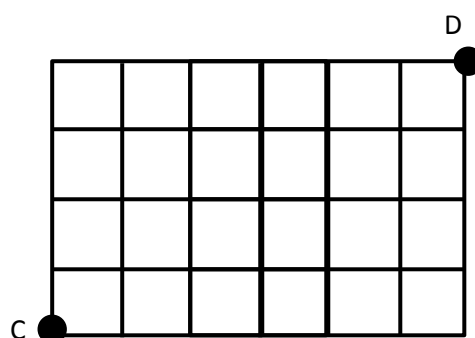
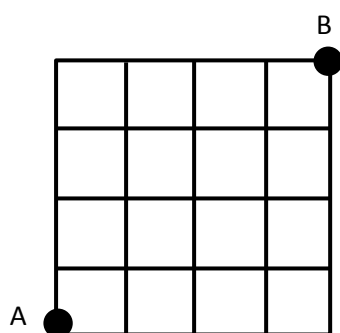
1. In how many ways can 7 red counters and 9 blue counters be arranged in a line?
2. In how many different ways can the letters ANNANNA be arranged?
3. In how many different ways can the letters BANANA be arranged?

Hint: First, figure out how many ways there are to place the As. Now ask yourself: How many spaces are left to put the Ns into? (Careful – there aren't 6 any more!) Now, how many ways can you put the 2 Ns into those spaces? Finally, the B will go into whatever space is left.

4. DNA sequences are written as lists, using the letters G, A, T, and C (hence the name of the film *GATTACA*). How many DNA sequences are there:
 - (i) That just use 6 Gs and 7As?
 - (ii) That use 11 Ts and 6 As?
 - (iii) That use 5 Gs, 5 Ts, and 5 As? (This is like Q3.)
 - (iv) That use 7 Gs, 6Ts and 3Cs?
 - (v) That use 4 of each letter?



5. (i) In how many ways can a driver get from A to B, from C to D, and from E to F, in each of these cities, presuming that she can only go up and to the right?



- (ii) How many of these car journeys from E to F go through G? (Try breaking the journey into two parts, and treat each part separately.)
- (iii) Where would you place G so that the number of journeys from E to F that go through G is as great as possible?
- (iv) In how many ways can you get from A to B, if you're also allowed to go left?
- (v)* How many different car journeys of length 12 can you take from C to D, if you're allowed to move right, left, or up?

Picking from Groups

6. 5 boys and 4 girls are playing doctors and nurses. They have 2 nurses' costumes and 2 doctors' costumes. In how many ways can they pick who gets to play if:
- (i) They don't care who wears what costumes?
 - (ii) The boys must wear the doctors' costumes and the girls must wear the nurses'?
7. Coach Carter must select a team of 3 attackers and 4 defenders from a panel of 7 attackers and 10 defenders.
- (i) In how many ways can he do this?
 - (ii) 2 of his attackers are injured, so they can't play. Now how many ways can he do it?
 - (iii) 2 attackers are injured, and 1 of the defenders is the captain, so he must start. Now how many ways are there for him to pick his team?

Sober or blotto, remember the Lotto!

8. For the Lotto, you pick any 6 numbers between 1 and 45. At each draw 6 Winning Numbers are drawn out. Once these numbers are drawn out, a Bonus Number is drawn from the remaining 39 numbers.



- (i) In how many different ways can the 6 Winning Numbers be drawn out?
 - (ii) In how many ways can the 6 Winning Numbers and the Bonus Number be drawn?
- Try to work out in how many ways each of the following can happen – at the moment, we don't care about whether you match the Bonus Number or not.

	Event	Win	Number of Ways
(iii)	Match 6 Winning Numbers	Jackpot! (min. €2million)	
(iv)	Match 5 Winning Numbers	€25,000 if you match the Bonus Number as well	
(v)	Match 4 Winning Numbers	Depends on how many win	
(vi)	Match 3 Winning Numbers	Depends on how many win	

Now try to work out these – make sure that in each one you've taken into account all 6 of the numbers you have to choose!

	Event	Win	Number of Ways
(vii)	Match 5 Winning Numbers (WNs) and the Bonus Number (BN)	€25,000	
(viii)	Match 5 WNs but not the BN	Depends on how many win	
(ix)	Match 4 WNs and the BN	Depends on how many win	
(x)	Match 4 WNs but not the BN	Depends on how many win	
(xi)	Match 3 WNs and the BN	Depends on how many win	
(xii)	Match 3 WNs but not the BN	€5 (Cheapskates!)	



9. To play the Euromillions Lottery, you choose 5 numbers, from 1 to 50, and 2 Lucky Star numbers, from 1 to 11. At each draw, 5 Winning Numbers (WNS) and 2 Lucky Star Numbers (LSs) are drawn out.

(i) In how many different ways can the 5 WNs and the 2 LSs be drawn?

Now try to work out in how many ways each of the following can happen. Again, make sure that in each one you've taken into account all 7 of the numbers you have to choose! (The amounts of money are in sterling, because the UK site gives figures.)

	Event	Win (average)	Number of ways
(ii)	Match 5 WNs and 2 LSs	Jackpot! (min. €15 million)	
(iii)	Match 5 WNs and 1 LS	£211,823.60	
(iv)	Match 5 WNs and no LS	£35,303.90	
(v)	Match 4 WNs and 2 LSs	£2824.30	
(vi)	Match 4 WNs and 1 LS	£137.20	
(vii)	Match 4 WNs and no LS	£68.60	
(viii)	Match 3 WNs and 2 LSs	£40.10	
(ix)	Match 3 WNs and 1 LS	£12.80	

Solutions to Worksheet

1. There's a total of 16 counters, so in $16C7 = 11440$ (or $16C9 = 11440$) ways.
2. There are 7 letters, and 3 As, so in $7C3 = 35$ ways.
3. There are 6 letters in total, and 3 As, so there are $6C3$ ways to place the As. There are then 3 spaces left (as 3 of the 6 spaces now have As in them), and 2 Ns, so there are $3C2$ ways to place the Ns. The B will have to go in the only space left over.
So in total there are $6C3 \times 3C2 = 60$ ways to rearrange the letters.
4. (i) There are 13 letters in total, so $13C6 = 1716$ ways.
(ii) There are 17 letters in total, so $17C11 = 12376$ ways.
(iii) There are 15 letters in total, and 5 Gs, so these can be placed in $15C5 = 3003$ ways.
There are now 10 spaces left, and 5 Ts, so these can be placed in $10C5 = 252$ ways.
The As will have to go into whatever spaces are free, so they add nothing.
So in total there are $15C5 \times 10C5 = 756756$ ways to arrange them.
(iv) Similar to previous section. If we place Gs, then Ts, then Cs, we get:
 $16C7 \times 9C6 = 960960$ ways.
(v) 16 letters in total, 4 of each. So: $16C4 \times 12C4 \times 8C4 = 63063000$ ways.
5. (i) A to B: 4 Rs and 4 Us, so $8C4 = 70$ ways.
C to D: 6 Rs and 4 Us, so $10C6 = 210$ ways.
E to F: 4 Rs and 7 Us, so $11C4 = 330$ ways.
(ii) E to G: 3 Rs and 3 Us, so $6C3 = 20$ ways.
G to F: 1 R and 4 Us, so $5C1 = 5$ ways.
So E to G, and then G to F: $20 \times 5 = 100$ ways.
(iii) By trial and error, or maybe intuition, you place it either 1 street above E, or 1 street below F. In either case, you get $10C4 \times 1C1 = 210$ ways.
(iv) In this case there are an infinite number of possible routes, as long as we assume that you're allowed go over the same street more than once (it didn't say that you couldn't). From A, simply go right, the left, then right, then left, and so on, as many times as you like – the number of different journeys is now unbounded (though repetitive).
The question as to how many ways there are if you can't go on a street more than once is trickier – this is asking how many ways you can arrange R, L, and U, so that:
(a) There are 4 Rs and 4 Us.
(b) R and L never appear side by side.

- (c) At any given point in the sequence, there must be, up to that point, at least as many Rs and Ls. (This ensures that we don't take a left off the map!)

This might be left as a challenge for the more able students!

- (v)* Note that a car journey from C to D without lefts has length 10. So a journey of length 12 must have one left, and one extra right to make up for it. So we need to order 7 Rs, 4Us and 1 L. The only restriction now is that there must be an R to the left of the L in the ordering – otherwise we'll drive off the map. We can try doing this directly, or we could calculate all possible orderings, then subtract those that have the L to the left of all the Rs.

Total number of orderings: $12C7 \times 5C4 = 3960$.

Orderings that start "L...": $11C7 = 330$, as there are now 11 spaces left to fill, if the first space is taken by an L.

Orderings that start "UL...": $10C7 = 120$.

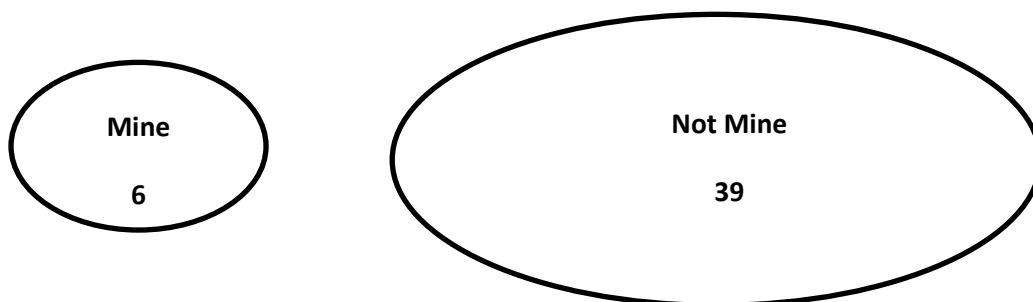
Orderings that start "UUL...": $9C7 = 36$.

Orderings that start "UUUL...": $8C7 = 8$.

Orderings that start "UUUUL...": $7C7 = 1$.

Admissible orderings: $3960 - (120 + 36 + 8 + 1) = 3795$ ways.

6. (i) If they don't care who wears what, then we need to pick 4 people from 9 to wear the costumes, i.e. $9C4 = 126$ ways.
- (ii) We need to pick 2 boys from 5: $5C2 = 10$ ways.
We also need to pick 2 girls from 4: $4C2 = 6$ ways.
So the total is $10 \times 6 = 60$ ways to pick the 4 kids to play.
7. (i) Pick 3 attackers from 7: $7C3 = 35$ ways.
Pick 4 defenders from 10: $10C4 = 210$ ways.
Total number of ways: $35 \times 210 = 7350$.
- (ii) Pick 3 attackers from just 5, and 4 defenders from 10: $5C3 \times 10C4 = 2100$ ways.
- (iii) As the captain must start, there are only 3 places left for defenders on the team, and 9 defenders who can fill these spaces. There are also 5 attackers, from which he must pick 3. So: $5C3 \times 9C3 = 840$ ways.
8. (i) Winning Numbers: $45C6 = 8145060$ ways.
- (ii) Bonus Number: 39 ways.
WNs and BN: $8145060 \times 39 = 317657340$ ways.
- We can split the numbers into Mine (i.e. the 6 numbers I picked) and Not Mine (the 39 I didn't pick):



- (iii) Match 6 WNs and 0 LNs: $6C6 \times 39C0 = 1$. Note: the only way the 6 winning numbers will match my numbers is if all 6 are matching, i.e. there is only 1 way!
- (iv) Match 5 WNs: $6C5 \times 39C1 = 234$ ways. Note that, if 5 of the WNs are from Mine, then 1 must be from Not Mine. This is the $39C1$.
- (v) Match 4 WNs: $6C4 \times 39C2 = 11115$ ways. Here, 4 of the WNs are from Mine, and 2 are from Not Mine.
- (vi) Match 3 WNs: $6C3 \times 39C3 = 182780$ ways.

For each of these, we need to take the Bonus Number (BN) into account as well.

- (vii) Match 5 WNs and 1 BNs: $6C5 \times 39C1 \times 1C1 = 234$ ways. Note that the first two terms are exactly as in (iv) above. The last term comes from the fact that there is only 1 of Mine free to match the BN.
 - (viii) Match 5 WNs and 0 BNs: $6C5 \times 39C1 \times 38C1 = 8892$ ways. Here, the BN can be any of the 39 Not Mine numbers, other than the 1 that's a WN.
 - (ix) Match 4 WNs and 1 BN: $6C4 \times 39C2 \times 2C1 = 22230$ ways. Here, the BN is chosen from the two Mine numbers that aren't WNs.
 - (x) Match 4 WNs and 0 BN: $6C4 \times 39C2 \times 37C1 = 411255$ ways. Here, the BN is chosen from the 37 Not Mine numbers that are not WNs.
 - (xi) Match 3 WNs and 1 BN: $6C3 \times 39C3 \times 3C1 = 548340$ ways. Here, the BN is chosen from the 3 Mine numbers that are not WNs.
 - (xii) Match 3 WNs and 0 BN: $6C3 \times 39C3 \times 36C1 = 6580080$ ways.
10. (i) 5 WNs from 50, and 2 LSs from 11: $50C5 \times 11C2 = 116531800$.

For each of the rest of the questions, we need to consider the Main Draw and the Luck Star Draw, as shown on below.



- (ii) Match 5 WNs and 2 LSs: $5C5 \times 45C0 \times 2C2 \times 9C0 = 1$.
Similar comment to Q8(iii).
- (iii) Match 5 WNs and 1 LS: $5C5 \times 45C0 \times 2C1 \times 9C1 = 18$.
- (iv) Match 5 WNs and 0 LSs: $5C5 \times 45C0 \times 2C0 \times 9C2 = 36$.
- (v) Match 4 WNs and 2 LSs: $5C4 \times 45C1 \times 2C2 \times 9C0 = 225$.
- (vi) Match 4 WNs and 1 LS: $5C4 \times 45C1 \times 2C1 \times 9C1 = 4050$.
- (vii) Match 4 WNs and 0 LSs: $5C4 \times 45C1 \times 2C0 \times 9C2 = 8100$.
- (viii) Match 3 WNs and 2 LSs: $5C3 \times 45C2 \times 2C2 \times 9C0 = 9900$.
- (ix) Match 3 WNs and 1 LS: $5C3 \times 45C2 \times 2C1 \times 9C1 = 178200$.

Week 12: Confounding Combinatorics – Calculating the Hands in Poker

Introduction

This is a Whole-Class lesson. It continues on from the previous two lessons, and looks at how to calculate the probability of being dealt the different hands in Poker, using nCr and the Fundamental Principle of Counting. Things get a little tricky here, so there's a chart with the deck of cards on it included in the Question Sheet for the students to refer to.

Resources

- Students should have their calculators.
- At least one pack of cards, for demonstration purposes. More packs for students, if desired.
- A copy of the Over The Odds Question Sheet, one per student.

Activities

Whole-Class Lesson:

- 1. Introducing Probability** (25 mins)
- 2. Probabilities of Poker** (25 mins)

Take-home Problem:

Finish worksheet.

Whole-Class Lesson

1. Introducing Probability (25 mins):

- (a) Discuss – what do probabilities range between? What does a probability of 0 mean? A probability of 1? Can we get examples of such events?
- (b) What's the probability of rolling a 5 on a die? What about an odd number? A prime? A number greater than 4?
- (c) So what rule could we use in general for probabilities? Aiming towards:

$$\text{Prob(Event } A \text{ happens)} = \frac{\text{Number of ways } A \text{ can happen}}{\text{Total number of possible outcomes}}$$

Relate this back to the die example.

- (d) Go through Q1 (a) to (c) with the class as an example of using probabilities. Students then attempt Q1 (d) to (g).
- (e) Go through Q2 (a) to (c) with the class. These are more like the Poker questions, as they are using nCr. Students then attempt Q2 (d) to (f). Note that the bottom lines are all the same – they are the number of different possible combinations that could be drawn out of the drum.

2. Probabilities of Poker (25 mins):

- (a) Introduce the structure of the deck of cards, and the different hands in Poker. It's worth spending a few minutes asking students to identify different hands - e.g. what hand is {A, A, 3, 9, 3}? – to get them used to the game. In particular, they need to spot that the example just given isn't a Pair, even though it might look like one – it's actually Two Pairs. This will be important in working out probabilities.
- (b) Go through the following as examples:
 - (i) Royal Flush – need to take into account each of the 4 suits.
 - (ii) Straight Flush – use 2 steps: 1st pick the cards, then the suit.
 - (iii) Trips – as well as picking the scoring cards (the 3 matching ones), you need to be careful in picking the other cards. If you have 3 Aces as your trips, neither of the other 2 cards can be Aces, nor can they be a matching pair (as then you'd have a Full House).
- (c) Students attempt the rest themselves. It's easier to start with the better hands, and work backwards towards the weaker hands.

Over the Odds – Calculating Probabilities

In probability, we use the word event to mean something that might happen – I see a red car today, or Ireland will win the World Cup, or I get a head when I flip a coin.

The most basic principle of probability is:

$$\text{Prob(Event } A \text{ happens)} = \frac{\text{Number of ways } A \text{ can happen}}{\text{Total number of possible outcomes}}$$

Using this idea, try to work out the following:

1. In Mastermind there are 5 different colours, and you need to guess a code of length 4 that has been generated randomly by a computer. If the code is not allowed to repeat colours:

- (a) How many different codes are possible?
- (b) How many of these start with Green?
- (c) What's the probability that the correct code starts with Green?








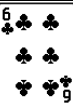

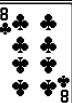












































Suppose now that the code *is* allowed repeat colours.

- (d) How many different codes are now possible?
 - (e) How many of those codes start with Blue?
 - (f) What's the probability that the correct code ends with Blue?
 - (g) What's the probability that the correct code starts with Blue, then Green?
2. Recall that to play the Euromillions Lottery, you choose 5 numbers, from 1 to 50, and 2 Lucky Star numbers, from 1 to 11. At each draw, 5 Winning Numbers (WNs) and 2 Lucky Star Numbers (LSs) are drawn out.
 - (a) In how many different ways can the 5 WN and the 2 LSs be drawn?
 Now try to work out the Number of Ways that each of these can happen, and the Probability that each one happens to you!

	Event	Number of Ways	Probability
(b)	Match 5 WN and 2 LSs		
(c)	Match 5 WN and 1 LS		
(d)	Match 5 WN and no LS		
(e)	Match 4 WN and 1 LS		
(f)	Match 3 WN and 1 LS		

Calculating the Hands in Poker

Layout of a Deck of Cards (no Jokers):

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs (Black)													
Hearts (Red)													
Spades (Black)													
Diamonds (Red)													

The next table gives the different hands in Poker. The aim is to figure out the probability of being dealt each one. You should write the probabilities as a fraction, and a decimal (the latter will make it easier to compare them).

The Different Hands in Poker:

Hand	Example	Probability	
1 Pair	<i>One pair of matching cards (e.g. 8, 8, 2, 4, J).</i>		
2 Pairs	<i>Two different pairs (e.g. 10, 10, Q, Q, 5).</i>		
Trips	<i>Three matching cards (e.g. 5, 5, 5, 3, A).</i>		
Straight	<i>A run of cards, not all of which are in the same suit. A straight can start or end on an ace (A, 2, 3, 4, 5, or 10, J, Q, K, A), but may not 'run over' an ace (e.g. Q, K, A, 2, 3).</i>		
Flush	<i>All cards of the same suit. They cannot form a straight.</i>		
Full House	<i>Three of one, and two of another (e.g. 5, 5, 5, Q, Q).</i>		
Poker	<i>All four of one card (e.g. A, A, A, A, 4).</i>		
Straight Flush	<i>A straight in a suit, but not to an ace.</i>		
Royal Flush	<i>A straight in a suit to an ace.</i>		

Solutions: Over the Odds – Calculating Probabilities

1. (a) There are 5 choices for the 1st colour, 4 for the 2nd colour (because you can't repeat colours, so one colour is now ruled out), 3 for the 3rd colour, and 2 for the last colour. So $5 \times 4 \times 3 \times 2 = 120$ different possible codes.
- (b) If the 1st colour is green, there are 4 choices for the 2nd colour, 3 for the 3rd colour, and 2 for the last colour. So $4 \times 3 \times 2 = 24$ different codes.
- (c) $\frac{24}{120} = \frac{1}{5}$. Which makes sense: if the first colour is picked randomly from 5 colours, there should be a $\frac{1}{5}$ chance that green is picked.
- (d) If you're allowed repeat colours, there are $5 \times 5 \times 5 \times 5 = 625$ different possible codes.
- (e) If blue is 1st, there are 5 choices for each subsequent colour, so there are $5 \times 5 \times 5 = 125$ different possible codes.
- (f) If the blue is last, there are 5 choices for each of the other colours. So there are $5 \times 5 \times 5 = 125$ different codes. So the chances of picking a code with blue last is $\frac{125}{625} = \frac{1}{5}$. Again, this should make sense, as did (c).
- (g) This can be done in $1 \times 1 \times 5 \times 5 = 25$ different ways. So the probability of it happening is $\frac{25}{625} = \frac{1}{25}$.
2. (a) There are $50C5$ ways of picking the 5 WNs, and $11C2$ ways of picking the 2 LSs. So there are $50C5 \times 11C2 = 116531000$ different possible combinations. This is going to be the bottom line in all of the following probabilities, as it is the total number of possible outcomes.

Using the answers from last week we have:

(b) $P(5 \text{ WNs, } 2 \text{ LSs}) = \frac{1}{116531000} = 0.00000000858, \text{ or } 0.000000858\%.$

(c) $P(5 \text{ WNs, } 1 \text{ LS}) = \frac{18}{116531000} = 0.0000001545, \text{ or } 0.00001545\%.$

(d) $P(5 \text{ WNs, } 0 \text{ LSs}) = \frac{36}{116531000} = 0.000000309, \text{ or } 0.0000309\%.$

(e) $P(4 \text{ WNs, } 1 \text{ LS}) = \frac{4050}{116531000} = 0.00003475, \text{ or } 0.003475\%.$

(f) We didn't do this one last week. In the Main Draw there are $5C3$ ways of picking which of the numbers in Mine are WNs, and $45C2$ ways of picking which of the numbers in Not Mine are WNs. In the Lucky Star Draw there are $2C1$ ways of picking which of the numbers in Mine are WNs, and $9C1$ ways of picking which of the numbers in Not Mine are WNs. So in total there are $5C3 \times 45C2 \times 2C1 \times 9C1 = 4178200$ ways. This gives a probability of $\frac{4178200}{116531000} = 0.03585, \text{ or } 3.585\%.$

Solutions: Calculating the Hands in Poker

We will start with the most restrictive hand – the Royal Flush – and work backwards. We will see that the hands get more likely the further we work back the list – there more than a 40% chance that you get dealt at least a pair.

Royal Flush:

There are only four different royal flushes, one in each suit.

$$P(\text{Royal Flush}) = \frac{4}{2598960} = 0.0000015$$

Straight Flush:

Each suit has ten flushes. However, as one of these is a royal flush, then each suit has 9 non-royal straight flushes. As there are four suits:

$$P(\text{Straight Flush}) = \frac{4 \times 9}{52C5} = \frac{36}{2598960} = 0.000014$$

Poker:

There are thirteen different ways to pick the four-of-a-kind (i.e. will I have four Aces, four 2s, etc). There are then 48 ways to pick the remaining, non-matching, card.

$$P(\text{Poker}) = \frac{13 \times 48}{52C5} = \frac{624}{2598960} = 0.00024$$

Full House:

Here, we will first pick the pair, then the trip. For the pair, there are 13 different choices for what number (or picture) the pair will be of. Once we've selected a number, there are $4C2$ ways of selecting what particular pair to use.

For the trip, there are now 12 different choices for the number (one of the possible choices already being used for the pair). Once we've selected a number, there are $4C3$ ways of selecting what particular trip to use.

So the probability of being dealt a full house is:

$$P(\text{Full House}) = \frac{(13 \times 4C2) \times (12 \times 4C3)}{52C5} = \frac{3744}{2598960} = 0.00144$$

Flush:

In any given suit, there are $13C5$ ways to pick out five different cards, and there are four different suits. However, as we don't want to count the 40 possible straight flushes (including the royal ones), we get:

$$P(\text{Flush}) = \frac{(4 \times 13C5) - 40}{52C5} = \frac{5108}{2598960} = 0.001965$$

Straight:

There are ten different starting cards for a straight (ace to 10), and for each card in the straight, there are four different possibilities. However, we again need to exclude the 40 straight flushes, so we get:

$$P(\text{Straight}) = \frac{(10 \times 4 \times 4 \times 4 \times 4 \times 4) - 40}{52C5} = \frac{10200}{2598960} = 0.0039$$

Trips:

First of all, there are thirteen different choices for the trip card, and, once that's chosen, there are $4C3$ ways of picking the three cards to use. There are then twelve different numbers (and pictures) left to choose from. This can be done in $12C2$ ways. Once these are decided upon, there are four ways to choose the first, and four to choose the second. We get the same result in this case:

$$P(\text{Trips}) = \frac{(13 \times 4C3) \times (12C2 \times 4 \times 4)}{52C5} = \frac{54912}{2598960} = 0.02113$$

2 Pairs:

There are $13C2$ ways of picking which two ranks will form the two pairs (e.g. Queens and sevens, or eights and twos), and then $4C2$ ways of choosing each particular pair. Finally, there are 44 ways to pick the final, non-scoring card:

$$P(2 \text{ Pairs}) = \frac{(13C2 \times 4C2 \times 4C2) \times 44}{52C5} = \frac{123552}{2598960} = 0.0475$$

1 Pair:

There are $13 \times 4C2$ ways of picking the pair. Once the pair is picked, there are $12C3$ ways to pick the numbers of the remaining cards. There are then four ways to pick the suit of each non-scoring card:

$$P(1 \text{ Pair}) = \frac{(13 \times 4C2) \times [12C3 \times 4 \times 4 \times 4]}{52C5} = \frac{1098240}{2598960} = 0.42257$$

Week 13: Confirming Combinatorics – Repeated Elements and Stars & Bars

Introduction

This is a Whole-Class lesson. It is the fourth, and final, combinatorics lesson. Students meet two new applications of nCr and the Fundamental Principle: repeated elements, e.g. how many ways can you rearrange the letters MISSISSIPPI (Section 1), which uses the same idea as the DNA sequencing question from the Lesson 13 but expands it a little; and Stars and Bars (Sections 2 to 4), a very useful technique for dealing with grouping a list of elements.

Resources

- Students should have their calculators.
- Counters for Activity 2 – Stars and Bars.
- A copy of the Question Sheet, one per student.

Activities

Whole-Class Lesson:

- 1. Recap & Repeated Elements (10 mins)**
- 2. Introducing Stars and Bars (15 mins)**
- 3. Stars and Bars with Minimums (10 mins)**
- 4. Stars and Bars with Maximums (15 mins)**

Take-home Problem:

Finish worksheet.

Whole-Class Lesson

1. Recap & Repeated Elements (10 mins):

- (a) Students try to work out how many ways the letters in the word XOXOXO can be arranged – we covered this two weeks ago.
- (b) Students try to work out how many ways the letters in the words MISSISSIPPI and COMMITTEE can be arranged – in each case, place the repeated letters before placing the unrepeatable ones.
- (c) Students try Questions 1 and 2 on the Question Sheet.

2. Introducing Stars and Bars (15 mins):

- (a) Suppose we have 10 sweets to share between Paul, Quentin and Ralph. In how many ways can we do this? Give students counters and let them try the problem for a few minutes. If they're finished, ask: what if we needed to share 100 sweets between 4 children?
- (b) We will approach this problem in a slightly different way:

Step 1: Lay out all of the 10 sweets in a row, represented by these stars:

* * * * *

Step 2: In order to divide up the sweets, put a bar between P's and Q's sweets, and a bar between Q's and R's. E.g., if $P=3$, $Q=6$, $R=1$:

* * * | * * * * * | *

We need to make sure that we can represent any distribution in this way – challenge students to represent, e.g., if $P=0$, in this manner. Ask them also to come up with unusual distributions, and make sure they satisfy themselves that this method can represent any outcome before continuing.

Step 3: Note that now there are 12 things that we are looking to arrange – 10 stars and 2 bars. So the question now becomes: in how many ways can we pick 2 of the 12 things to be our bars? This can be done in $12C2 = 66$ ways.

Note that you need one less bar than groups – e.g. to split something into 6 groups you'll need 5 bars. Here we had 3 groups, so 2 bars.

- (c) Try Questions 3 to 5 on the Question Sheet now, in the same way.

3. Stars and Bars with Minimums (10 mins):

- (a) Go through Question 6 on the Question Sheet – this involves giving certain people a minimum amount, before continuing on as usual. So, for instance, in

Q6(a), start by giving Noel €10. There is now €90 to be shared between 3 people – proceed as above.

(b) Students now try Questions 7 to 9 on the Question Sheet.

4. Stars and Bars with Maximums (15 mins):

(a) Go through Question 10 on the Question Sheet – this involves giving certain people a maximum amount: you get the overall amount, and subtract off those cases where the person has more than the maximum. See solution below.

(b) Students now try Questions 11 to 13 on the Question Sheet.

Repeated Elements and Stars and Bars

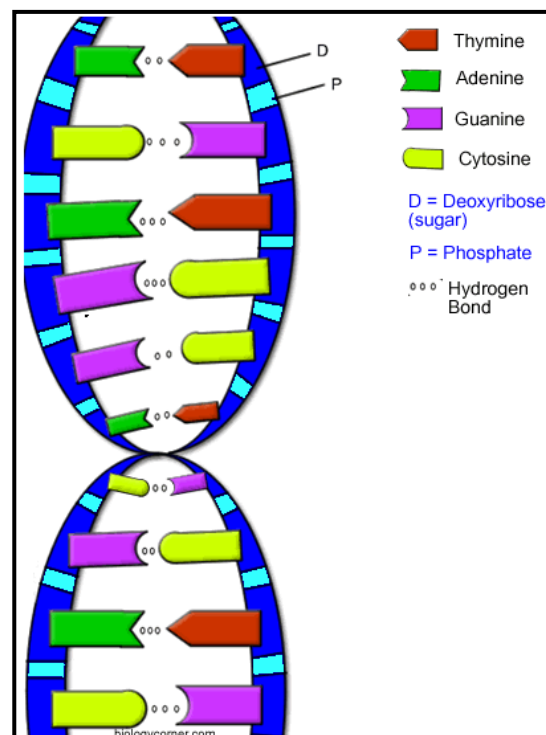
Note: In each of the following, we're only interested in how many things we have in each group – so, for example, in Q7 we don't care *who* the people in each of the lanes are, just how many people are in the Fast lane, how many in the Medium lane, and how many in the Slow lane.

1. How many different ways are there of arranging the letters in each of these:

- (a) ALABAMA
- (b) RGBRGBRGB
- (c) BANANARAMA

2. Recall that DNA sequences are formed from the letters G, A, T, and C.

- (a) How many different DNA sequences of length 16 are possible?
- (b) How many different DNA sequences of length 16 have an equal number of each letter?
- (c) If a DNA sequence of length 16 is picked at random, find the probability that:
 - (i) It has an equal number of each letter?
 - (ii) It has 10 As, and an equal number of each of the other letters?



3. John, Kyle, and Leanne are sharing 50 sweets.

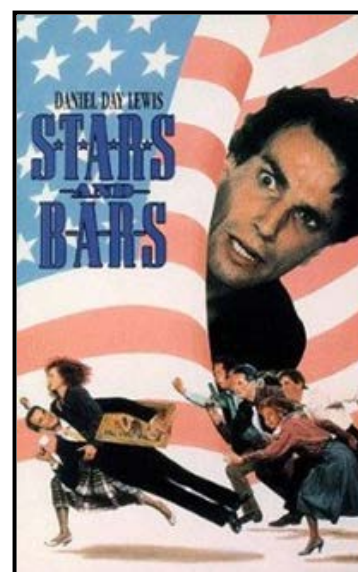
- (a) In how many ways can they do this?
- (b) Maurice joins them, bringing 10 sweets of his own that he contributes to the total. Now in how many ways can they share the sweets?

4. There are 12 people waiting to pay in a shop, and there are 4 checkouts open. In how many ways can the people be distributed between the 4 checkouts?

5. Suppose $x + y + z = 80$, where x , y , and z are all Natural Numbers. How many different ways are there to pick x , y , and z ? (Include 0 as a Natural Number.)

6. Noel, Oliver and Pauline have €100 in €1 coins that they are going to share between themselves. In how many ways can this be done if:

- (a) Noel must get at least €10?
- (b) Everyone must get at least €15?



7. There are 3 swimming lanes: Fast, Medium, and Slow. In how many ways can 10 swimmers be distributed between the lanes:
- (a) If there are no restrictions?
 - (b) If there must be at least 1 person per lane?
 - (c) If there must be at least 2 people in the slow lane, and at least 1 in each of the others?
8. Suppose $a + b + c + d = 55$, where $a, b, c, d \in \mathbb{N}$. In how many ways can the values of a, b, c , and d be chosen if:
- (a) There are no restrictions?
 - (b) $a > 5$ and $d \geq 10$?
 - (c) $b > 10, c > 5, d \geq 15$?
 - (d) $b = 16$, and the rest are all at least 5?
9. I'm sharing 30 sweets with 4 other people. In how many ways can the sweets be divided out between all of us if:
- (a) There are no restrictions?
 - (b) I get at least 6 sweets?
- Using these two answers:
- (c) Calculate the probability that, if the sweets are distributed at random, I'll get at least 6 of them.
- John joins our group, bringing with him 6 sweets that he adds to the total.
- (d) Calculate the probability that, if the sweets are now distributed at random, I'll get at least 6 of them. Have my chances changed?
10. 5 people – Am, Not, Making, This and Up – share a prize of €200. In how many ways can the prize be divided up, in whole number amounts, if Am can't get more than €20?
11. Achmed, Eoin, Irene, Orla and Ursula are sharing 40 collectible coins. In how many ways the share them, if Eoin can get at most 10, and Ursula gets at least 12?
12. 10 people arrive at a bank, where there are 4 banklink machines. In how many ways can the people queue for the machines if one particular machine can't have more than 4 people queuing at it?
13. 4 people have to perform 60 hours of community service between them. In how many ways can they allocate the hours so that everyone serves at least 10 hours, and one particular person can't serve more than 20 hours?

Solutions

1. (a) There are 7 spaces to fill. There are $7C4 = 35$ ways to place the A's. There are then 3 ways to place the L, 2 ways to place the B, and 1 way to place the M. That gives a total of $35 \times 3 \times 2 \times 1 = 210$ ways in total.
- (b) There are 12 spaces to fill. There are $12C4 = 495$ ways to place the Rs, then $8C4 = 70$ ways to place the G, and $4C4 = 1$ way to place the B. So the total is $495 \times 70 \times 1 = 34650$ ways in total.
- (c) Placing the 5 A's, then 2 N's, then B, then R, then M, gives:
 $10C5 \times 5C2 \times 3C1 \times 2C1 \times 1C1 = 15120$ different ways.
2. (a) There are 4 choices for each letter, so $4^{16} = 4294967296$.
- (b) We have 16 spaces, and 4 of each letter. So there will be:
 $16C4 \times 12C4 \times 8C4 \times 4C4 = 63063000$ different DNA sequences.
- (c) (i) $\frac{63063000}{4294967296} = 0.0147$, or 1.47%.
- (ii) 10 As, 2 Gs, 2 Cs, and 2 Ts: Number of ways is
 $16C10 \times 6C2 \times 4C2 \times 2C2 = 720720$.
 So probability is $\frac{720720}{4294967296} = 0.000168$, or 0.0168%.

In each of the following, we will indicate the number of stars (*) and bars(|), and then calculate the number of ways to arrange them. Remember that if things are being shared into n groups, we need $n - 1$ bars.

3. (a) 3 people share 50 sweets: $50^*, 2|$: $52C2 = 1326$.
- (b) 4 people share 60 sweets: $60^*, 3|$: $63C3 = 39711$.
4. 12 people to be split into 4 groups: $12^*, 3|$: $15C3 = 455$.
5. If we allow the numbers be 0, there are 80 units to share between 3 groups:
 $80^*, 2|$: $82C2 = 3321$.
6. (a) Give Noel €10. Now there is €90 left, to be shared between 3 people:
 $90^*, 2|$: $92C2 = 4186$.
- (b) Give each person €15. Now there is €55 left, to be shared between 3 people:
 $55^*, 2|$: $57C2 = 1596$.
7. (a) 10 people, 3 groups: $10^*, 2|$: $12C2 = 66$.
- (b) Put 1 person in each lane. Now 7 people, 3 groups: $7^*, 2|$: $9C2 = 36$.
- (c) Put 2 in Slow lane, 1 in each of the others. Now 6 people, 3 lanes: $6^*, 2|$:
 $8C2 = 28$.
8. (a) 55 units, 4 groups: $55^*, 3|$: $58C3 = 30856$.
- (b) Give a 6 units and d 10 units. Then 39 units left, 4 groups: $39^*, 3|$:
 $42C3 = 11480$.
- (c) Give b 11 units, c 6 units, and d 15 units. So 23 units left, 4 groups: $23^*, 3|$:
 $26C3 = 2600$.

- (d) Give b 16 units, and 5 units to each of the others. Then 24 units left, being split into just 3 groups this time (we won't give b any more, as $b = 16$): $24^*, 2 \mid : 26C2 = 325$.
9. (a) 30 sweets, 5 people: $30^*, 4 \mid : 34C4 = 46376$.
 (b) Give myself 6. Now 24 sweets, 5 people: $24^*, 4 \mid : 28C4 = 20475$.
 (c) $\frac{20475}{46376} = 0.4415$, to 4 decimal places.
 (d) There are 36 sweets, to be shared among 6 people. In total, this can be done in $36^*, 5 \mid : 41C5 = 749398$ ways. If I take 6 myself, then there are 30 sweets, 6 people: $30^*, 5 \mid : 35C5 = 324632$ ways. So the probability I get at least 6 is: $\frac{324632}{749398} = 0.4332$, to 4 decimal places. So my chances have gone marginally down, from 44.15% to 43.32% – a drop of less than 1%.
10. First of all, find the total number of different ways in which the fund can be divided up. $200^*, 4 \mid : 204C4 = 70058751$ ways. Next, give Am €21. Now $179^*, 4 \mid : 183C4 = 45212895$ ways. As these are the ways we want to exclude, we subtract them from the total to get 24845856 permissible ways.
11. Ursula must get at least 12, so give her 12. Now, we first find the total number of ways in which the remaining coins can be divided up: $28^*, 4 \mid : 32C4 = 35960$ ways. Next, give Eoin 11 coins, on top of the 12 Ursula has: $17^*, 4 \mid : 21C4 = 5985$ ways. As these are the ways we want to exclude, we take them from the total to get 29975 permissible ways.
12. Ignoring restrictions, the total number of ways is: $10^*, 3 \mid : 13C3 = 286$ ways. Now give that one ATM 5 people: $5^*, 3 \mid : 8C3 = 56$ ways. As these are the ways we wish to exclude, we subtract, to get 230 permissible ways.
13. First of all, give everyone 10 hours, so that we have 20 left to distribute. The total number of different ways that these 20 can be allocated is: $20^*, 3 \mid : 23C3 = 1771$. Now, give that particular person another 11 hours, so that he has 21. The number of different ways that the remaining 9 hours can be allocated in is given by: $9^*, 3 \mid : 12C3 = 220$. So the number of permissible ways is the difference between these two, i.e. 1551 different ways.

Week 14: Nasty Number Tricks and Devious Divisibility Tests.

Introduction

This is a Whole-Class lesson. Students meet 2 number tricks, which can be shown to always work using basic algebra. They then investigate how to check if a given number is divisible by 2, 3, 4, 5, 6, 8, 9, 10, or 11.

Resources

- Calculators for first number trick.
- One copy of the Nasty Number Tricks Worksheet for each student.

Activities

Whole-Class Lesson:

1. Left With 10 (10 mins)
2. Left with 9 (10 mins)
3. Divisibility Tests (30 mins)

Take-home Problem:

- (i) What numbers from 12 to 25 can you find simple divisibility tests for?
- (ii) Can you test to see if a number is divisible by 11?

Whole-Class Lesson

1. Left With 10 (10 mins):

Go through the number trick with the students – encourage them to use negative numbers, fractions, etc.

Once they've seen that it works, use x to show that it will always work. Those who catch on quickly could devise their own tricks – you could encourage them to be as elaborate as they like!

Finally, ask students what happens if they start with 0 – at what point does the process break down? This could lead to a discussion of why you can't divide by 0.

2. Left with 9 (10 mins):

Go through the number trick with the students – again, encourage them to try different numbers (as long as they're two-digit, with different digits). This time, because they're doing things with digits, they will need to write the number as $10x + y$, where x is the first digit and y the second. (You may need a few concrete examples here). Now go through the steps, and you should end with 9, as required.

Again, if we allow the digits to be the same, we end up dividing by 0.

3. Divisibility Tests (30 mins):

Go through the different tests with them – they should be able to get the tests for 10, 2, 5, 4, and 8 without too much difficulty. To make these clearer, it might be useful to write out the different digits, along with their place value – this will certainly be useful for 3 and 9.

Any proofs not covered could be left for homework, though it would be important that the proof for 3 be covered.

Nasty Number Tricks & Devious Divisibility

1. Left With 10

Go through the following steps, and see what answer you get. Then try it with a different starting number, and see what answer you get. In fact you can try it with any starting number you like (except 0), even fractions or negative numbers – you should get the same answer each time!

- Step 1** Pick any number, except 0.
- Step 2** Add 5 to the number.
- Step 3** Square your answer.
- Step 4** Subtract 25.
- Step 5** Divide by your original number.
- Step 6** Subtract your original number.
And the number you are left with is...?



My internet was broken so I had to use Clipart. Hence the toaster. Clipart doesn't give a lot of choice...

- (a) Can you figure out why this always works? You could go through the steps, using x instead of a number for step 1.
- (b) Try to create your own number trick that will always leave the same answer no matter what the starting number. Show that it works by using x instead of a starting number!
- (c) Why did we rule out 0 at the start? Try doing the trick with 0, and see where you run into difficulty. If you do it on a calculator, you will get an error at one of the steps. Why is this?

2. Left with 9

Again, you will get the same answer here no matter what number you start with, as long as you follow the steps:

- Step 1** Take any 2-digit number, with different digits.
- Step 2** Reverse the digits, and take the smaller of the 2 numbers from the bigger one.
- Step 3** Divide your answer by the difference between the 2 digits in your original number. And the number you are left with is...?

- (a) Try the trick with different initial numbers to see that it works.
- (b) This time we can't use x as our initial number because we're interested in the *digits* – so what should we use?
- (c) Again, the question ruled out certain numbers – why couldn't we use 2-digit numbers where the 2 digits were the same? At what step would the process break down?

3. Divisibility Tests

We're going to look now at *divisibility* – whether or not one number divides into another, without a remainder. For example, we say that 35:

- *is divisible by 5*, because 5 goes in 7 times with no remainder.
- *is not divisible by 3*, because 3 goes in 11 times, with a remainder of 2.

Write down a very big number, like 3140123412. We are going to see if this number is divisible by 2, 3, 4, 5, 6, 8, 9, 10, or 11. We'll start with the easiest first!

- (a) Is this number divisible by 10? How do we know?
- (b) We can tell a lot from the last digit – what other numbers can we rule in or out from just the last digit? Can you explain why these tests work?
- (c) How do we tell if the number is divisible by 4? How many digits do we need to look at? Why?
- (d) What about being divisible by 8? How many digits would we need to check for that? Why?

These next ones are a bit trickier – we'll need to use all of the digits here!

- (e) How would we check if the number is divisible by 3? It'll help if we write out each of the digits separately, with their place value. E.g. if we had 375, we'd write it as $(3 \times 100) + (7 \times 10) + (5 \times 1)$. Try the same with the big number above. Now is there anything you can do with the 10, 100, etc?
- (f) How would we check if the number is divisible by 9? It's much the same process as for 3.
- (g) How about checking for 6? Would any of our previous tests be relevant?

Take-home Problems:

- (i) What numbers from 12 to 25 can you find simple divisibility tests for?
- (ii) Can you test to see if a number is divisible by 11?

Solutions

1. Left With 10

(a) Using x as the original number, the steps in the trick yield:

(i) x

(ii) $x + 5$

(iii) $(x + 5)^2$, which can be expanded to give $x^2 + 10x + 25$.

(iv) $x^2 + 10x$

(v) $x + 10$

(vi) x

(c) This is about seeing why you can't divide by 0. It is useful here to go back to basics and ask: what do we mean by $10 \div 2$? What we mean is: how many 2s do I need to add together to get 10 as an answer? So $1 \div 2 = 5$, because $2 + 2 + 2 + 2 + 2 = 10$. (We could also think of it as meaning: if I divided 10 into 2 equal parts, how big would each part be? However, I think that our way will make more sense when we come to 0.) Similarly, $4 \div \frac{1}{3}$ means: how many $\frac{1}{3}$ s do I need to add together to get 4 as an answer? As I would need to add 12 of them to get 4, the answer is 12.

Now consider something like $1 \div 0$. This is asking us: how many 0s do I need to add together to get 1? The answer is: there's no amount of 0s I could add to get 1. So $1 \div 0$ doesn't make any sense. Similarly, $x \div 0$ doesn't make any sense for any non-zero x . So we say that $x \div 0$ is not defined, if $x \neq 0$.

Finally, what does $0 \div 0$ mean? Well, it means: how many 0s do I need to add together to get 0? Here, the problem isn't that there's no answer – it's that there are too many answers! Any number you like could be the answer – for instance, two 0s add to give 0, as do five 0s, or a million 0s. Because of this confusion, we say that $0 \div 0$ is also undefined – if I were to write it down, you would have no idea what number I meant!

2. Left With 9

(b) Call the number $10x + y$. We'll assume that x is greater than y – because we'll be reversing the digits, it doesn't really matter which digit is bigger. The steps then yield:

(i) $10x + y$

(ii) Reversing the digits gives $10y + x$ (do some concrete examples to show this), and subtracting gives $10x + y - (10y + x) = 9x - 9y$.

- (iii) The difference between x and y is $x - y$. Writing the answer to the last step as $9(x - y)$ gives an answer here of $\frac{9(x-y)}{x-y} = 9$.

3. Divisibility Tests

(a) From the last digit, you can check if a number is divisible by:

&(b) 10: ends in 0.

2: ends in 2, 4, 6, 8, or 0.

5: ends in 5 or 0.

The reason for this is that the value of every digit other than the last digit is some multiple of 10, and so is divisible by 10, 5 and 2. Thus we only need check the last digit to see if these divide into the number.

(c) 4 divides into 100. The value of each digit, apart from the last 2 digits, is some multiple of 100, and so is divisible by 4. So we need only check to see that 4 divides into the last 2 digits. In the example, 4 divides into 12, so 4 divides into the whole number.

(d) Similarly, 8 divides into 1,000. The value of each digit, apart from the last 3, is some multiple of 1,000, and so is divisible by 8. So we need only check to see that 8 divides into the last 3 digits. In the example, 8 doesn't divide into 412, so 8 doesn't divide into the whole number.

(e) To see if 3140123412 is divisible by 3, we need to split up the digits – let's start with the digit in the units place and work from there:

$$(2 \times 1) + (1 \times 10) + (4 \times 10^2) + (3 \times 10^3) + \dots + (3 \times 10^9)$$

Now split up each of the powers of 10 as follows:

$$(2 \times 1) + (1 \times 9 + 1 \times 1) + (4 \times 99 + 4 \times 1) + (3 \times 999 + 3 \times 1) + \dots \\ \dots + (3 \times 999999999 + 3 \times 1)$$

Because 3 divides into 9, 99, 999, etc., then we can ignore each term with 9, 99, 999, etc. in it when seeing what the remainder is when we divide by 3.

So we are left looking at the number:

$$(2 \times 1) + (1 \times 1) + (4 \times 1) + (3 \times 1) + \dots + (3 \times 1)$$

which is simply the sum of the digits of the number.

So if we want to check if 3 divides into a number, we need only check if 3 divides into the sum of the digits of the number. Moreover, if 3 doesn't divide in, then whatever the remainder when we divide 3 into the sum of the digits will be the same as the remainder when we divide 3 into the number itself.

In this case, the sum of the digits is 21. As 3 divides into this, it will divide into the whole number.

- (f) To see if a number is divisible by 6, check if it is even (i.e. divisible by 2) and divisible by 3. If it is both of these, then it must be divisible by 6. So our number here is divisible by 6.
- (g) To check if a number is divisible by 9, you can use the same method as for divisibility by 3 – add the digits, and see if they are divisible by 9. This works because 9 divides into 9, 99, 999, etc.

So our number is not divisible by 9 – the sum of the digits is 21, which 9 does not divide into. Moreover, when we divide the sum of the digits by 9 we get a remainder of 3 – this is the same remainder we would get if we divided the original number by 9.

Take-home Problem

- (i) Students should get at least the following:
- 12: Check if it's divisible by 4 and by 3.
 - 15: Check if it's divisible by 5 and by 3.
 - 16: Ignore everything but the last 4 digits. Does 16 divide this number?
 - 18: Check if it's divisible by 2 and 9.
 - 20: Remove the last digit. Check if what's left is divisible by 2.
 - 22: Check if it's divisible by 2 and 11 (presuming that you can check for 11).
 - 24: Check if it's divisible by 8 and 3.
 - 25: Remove everything but the last 2 digits. Does 25 divide this number?
- (ii) This is a lot easier if you introduce modular arithmetic (which is one of the things for which this lesson is setting some groundwork, or at least motivating). Using our current method, we could observe that we can write 3140123412 as:
- $$(2 \times 1) + (1 \times 11 - 1 \times 1) + (4 \times 99 + 4 \times 1) + (3 \times 1001 - 3 \times 1) + \dots + (3 \times 1000000001 - 3 \times 1)$$
- The trick here is to observe that 11 divides 11, 99, 1001, 9999, etc. So in this case, to see if 11 divides 3140123412, we need to look at:
- $$2 - 1 + 4 - 3 + 2 - 1 + 0 - 4 + 1 - 3$$
- This is equal to -3 , which is not divisible by 11, so 11 does not divide our original number.

Final note: To see that 11 divides 11, 99, 1001, 9999, etc., observe that:

- For an even number of 9s, we have $9999 \dots 99 = 11 \times 9 \times 1010 \dots 01$. There are as many 1s in $1010 \dots 01$ as there are pairs of 9s in $9999 \dots 99$. So 11 divides $9999 \dots 99$ in this case.
- For an even number of 0s, we have $100 \dots 001 = 11 + 99 \dots 990$. In this case, $99 \dots 990$ has an even number of digits, and so is divisible by 11 from the previous case. Thus $100 \dots 001$ is also divisible by 11.

Week 15: Nefarious Number Tricks: 1089, and Why A Square Number Can Never End In 7.

Introduction

This is a Whole-Class lesson. Students meet the 1089 trick, and look at what remainders a perfect square can leave, when divided by a given number (4 and 10 are used in the lesson).

Resources

- None needed.

Activities

Whole-Class Lesson:

1. 1089 (15 mins)
2. Remainder When Dividing by 4 (20 mins)
3. The Last Digit of a Square (15 mins)

Take-home Problem:

Find the remainder of $1^2 + 2^2 + 3^2 + \dots + 50^2$ when divided by 5.

Find also the last digit of $1^2 + 2^2 + 3^2 + \dots + 100^2$.

Whole-Class Lesson

1. 1089 (15 mins):

Go through the following trick with students:

- (i) Pick any 3-digit number, where the first and last digits differ by 2 or more.
- (ii) Reverse the number, and subtract the smaller of the two numbers from the larger one.
- (iii) Reverse the result, and add. The answer will be ... 1089.

Allow students to try it with different numbers, to make sure that it always works. Then see if they can construct how it works – this time, there is a trick half way through the proof.

2. Remainder When Dividing Squares By 4 (20 mins):

Ask students if they can find, without using a calculator, what the remainder of the following will be when divided by 4:

$$1^2 + 2^2 + 3^2 + \dots + 20^2$$

Look for a pattern here – first of all can we find one, then can we explain or prove it?

3. The Last Digit of a Square (15 mins):

Ask the students to find a perfect square that ends in 7. Allow them time to try it – they won't manage to find one.

Challenge the students to find, with proof, what digits a square can end in – how does this relate to the last question? Could you pursue a similar approach?

- $4n$
- $4n + 1$
- $4n + 2$
- $4n + 3$

Squaring each of the above gives, respectively:

- $16n^2$
- $16n^2 + 8n + 1$
- $16n^2 + 16n + 4$
- $16n^2 + 24n + 9$

Of these, 4 clearly divides into each term of the first and third.

For the other two, 4 divides into the first 2 terms of each. However, the constant term in each leaves a remainder of 1 when divided by 4, so the whole number will leave a remainder of 1 when divided by 4.

3. The Last Digit of a Square

This is just like the last question, except instead of $4n$, $4n + 1$, etc, you use $10n$, $10n + 1$, $10n + 2$, etc.

It's not a bad exercise to square out all of these numbers – it might challenge some students into seeing that, each time, they get $(10n + k)^2 = 100n^2 + 20nk + k^2$. Here, 10 clearly divides into the first 2 terms, so we really only need look at the last digit of k^2 , for $0 \leq k \leq 9$.

The thing to emphasise is that, if we're interested in the last digit of the result, we only need look at the last digit of each number as we go along. This will be useful in the take-home problem.

Take-Home Problem

For the first part, the numbers $5n$, $5n + 1$, $5n + 2$, $5n + 3$, and $5n + 4$ will leave remainders of 0, 1, 4, 4, and 1, respectively, when they are squared. So, for instance, when $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ is divided by 5, it leaves a remainder of $1 + 4 + 4 + 1 + 0$, i.e. of 10. As this is divisible by 5, then the sum leaves a remainder of 0.

Similarly, the sum of any 5 consecutive squares will leave a remainder of 0 when divided by 5. The sum in the question is composed of 10 distinct such series, so it has a remainder of 0 when divided by 5.

The second part is similar – the remainders of the 10 squares are, beginning from $(10n)^2$, 0, 1, 4, 9, 6, 5, 6, 9, 4, and 1. These sum to 45, which leaves a remainder of 5 when divided by 10. (We're looking for the last digit, so we're dividing by 10.) The sum in the question can be separated into 10 such groups, each with a remainder of 5, so the total leaves a remainder of 50, i.e. of 0, when divided by 10.

Week 16: Gruesome Games – Symmetry

Introduction

This is a Whole-Class lesson. It is the first of a series of lessons involving games, and their solution. Students often find it difficult to ‘solve’ these games adequately – that is, first in identifying a winning strategy, and secondly in proving that it always works. As such, this is an interesting introduction to the idea of a proof in maths.

All of the games this week involve symmetric strategies.

Resources

- Coins or counters for Games 1 and 2.
- Square paper would be useful for Game 5.
- A copy of the Gruesome Games Activity Sheet, one per student.

Activities

Whole-Class Lesson:

The basic ideas are explained to the students:

- Each game is for 2 people (so if there’s more than 2 in a group, students need to form into 2 teams).
- In each case, there is a foolproof winning strategy. However, in some games the first player can always win, while in others the second player can always win. If neither player knows the winning strategy at the outset, then either player may win. So in each game, students should take turns going first.

Students play each game, in order. They do not move on to the next game until they have worked out how to solve the current one, i.e. worked out which player can always win, and how.

It would be useful, after students have played the first game for a while, to go through the solution, including a proof, i.e. an argument that convinces everyone that the strategy will always work, no matter what move the other player makes.

Take-home Problem:

Finish off games on Gruesome games Activity Sheet.

Gruesome Games

In each case, figure out who wins, and how!



Game 1

There are three piles of coins: one with 5 coins, one with 10 coins, and one with 15 coins.

At each turn, one player can choose one of the piles and divide it into two smaller piles. The loser is the player who cannot do this.

Game 2

There are two piles of coins, with 13 coins in each one.

At each turn, a player may take as many coins as they like, but only from one of the piles. The player removing the last coin wins.

What if there were two piles, one with 7 coins and one with 8 coins?

Game 3

Ten points are placed around a circle.

Players take turns joining two of the points with a line segment which does not cross a segment already drawn. (Lines can touch at a common point on the circle.) The player who cannot draw a line loses.

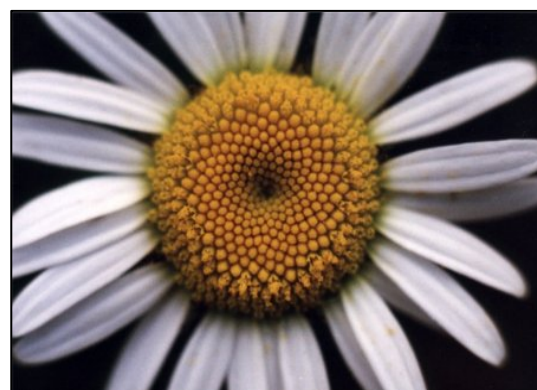
Game 4

A daisy has 12 petals.

Players take turns tearing off either a single petal, or two petals right next to each other.

The player who removes the last petal wins.

What if the daisy had only 11 petals?



Solutions

Game 1

There are three piles of coins: one with 5 coins, one with 10 coins, and one with 15 coins. At each turn, one player can choose one of the piles and divide it into two smaller piles. The loser is the player who cannot do this.

Each time you divide a pile, you create one extra pile. In order to create individual coins, the pile with 5 coins must be divided 4 times; the one with 10 coins, 9 times; and the one with 15 coins, 14 times. So 27 divisions must take place, i.e. the person going first will win.

Game 2

There are two piles of coins, with 13 coins in each one.

At each turn, a player may take as many coins as they like, but only from one of the piles.

The player removing the last coin wins.

However many coins the first player removes from a pile, the second player removes the same number from the other pile. This ensures that the second player will win.

What if there were two piles, one with 13 coins and one with 14 coins?

Here, the person going first removes one coin from the pile with 14. They now use the strategy for the original game, and win.

Game 3

Ten points are placed around a circle.

Players take turns joining two of the points with a line segment which does not cross a segment already drawn in. (Lines can touch at a common point on the circle.) The player who cannot draw a line loses.

The first player draws a line that has 4 points on either side of it. For the remainder of the game, whatever line the second player draws, the first player draws its image, under symmetry in the original line. This guarantees that the first player wins.

Game 4

A daisy has 12 petals.

Players take turns tearing off either a single petal, or two petals right next to each other. The player who removes the last petal wins.

The second player can win. No matter how the first player chooses their petals, the second player can choose theirs so that there are two identical rows of petals left on the flower. They then follow a symmetric strategy.

It makes no difference if the daisy only has 11 petals – the second player can do just as they did in the original case.

Week 17: Ghoulish Games – Working From The Endgame

Introduction

This is a Whole-Class lesson. It is the second of a series of lessons involving games. This week, a strategy called *Working from the Endgame* is introduced.

Resources

- Chessboards and pieces, one per group.
- A copy of the Ghoulish Games Activity Sheet, one per student.
- Matches for Game 6 (Nim).
- Checkers, or just square paper, for Game 7.

Activities

Whole-Class Lesson:

Once again, students are split into pairs (or groups, with each group divided into 2 teams).

Students initially play Games 1 and 2 themselves – if they are having trouble, the teacher should point out that they might try symmetric strategies, like last week.

Students then try Games 3 and 4, to see if they can solve them. Once they have (or are stuck), teacher introduces idea of Working from the Endgame.

Students use this idea to solve the remaining games. They do not move on to the next game until they have worked out how to solve the current one, i.e. worked out which player can always win, and how.

Game 6 was met in Week 1. We now approach it more systematically, and see its connection to the other games here.

Note: It's assumed that students know how a rook and a bishop move.

Take-home Problem:

Finish off games on Ghoulish Games Activity Sheet.

Ghoulish Games

In each case, figure out who wins, and how!

Game 1

Two players take turns placing rooks (i.e. castles) on a chessboard, so that they cannot capture each other. The loser is the player who cannot place a rook.

Game 2

Two players take turns placing bishops on the squares of a chessboard, so that they cannot capture each other. (They can be placed on squares of any colour.) The loser is the player who cannot place a bishop.

Game 3

On a chessboard, a rook stands on a1, the bottom left-hand square.

Players take turns moving the rook as many squares as they want, either horizontally to the right or vertically upward.

The player who can place the rook on the top right-hand square (h8) wins.

Game 4

On a chessboard, a king stands on a1.

Players take turns moving the king either one square upwards, to the right, or along a diagonal going up and to the right. The player who can place the king on h8 wins.

Game 5

A queen stands on square b1. Players take turns moving the queen any number of squares upwards, to the right, or along a diagonal going up and to the right.

The player who can place the queen on h8 wins.

Game 6 (Nim)

A bundle of 32 matchsticks is put in the middle of the table.

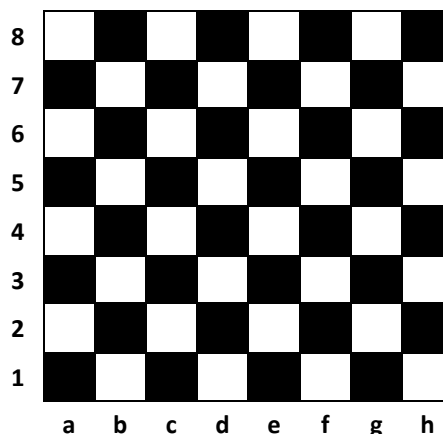
Players take turns to remove one, two, or three sticks at a time. The winner is the person who removes the last stick.

Game 7

A checker is placed at each end of a strip consisting of 1×20 squares.

Players take turns moving either checker in the direction of the other, by 1 or 2 squares. Neither checker can jump the other. The player who cannot move loses.

What would happen if it was a strip of 1×21 squares?



Solutions

Game 1

Two players take turns placing rooks (i.e. castles) on a chessboard, so that they cannot capture each other. The loser is the player who cannot place a rook.

Second player wins, no matter how the game is played. Each rook rules out the row and column that it's on. As each rook must rule out a different row (and column) from each previous rook, then 7 rows and 7 columns have been ruled out after 7 goes, leaving one square free on which the 8th rook can be placed. Now all rows and columns are ruled out.

Game 2

Two players take turns placing bishops on the squares of a chessboard, so that they cannot capture each other. (They can be placed on squares of any colour.) The loser is the player who cannot place a bishop.

Second player wins, if, each time the first player places a bishop, the second player places one in a position that's symmetric about an axis running straight down the middle of the board, between the 4th and 5th columns. First of all, at the end of each of the second player's turns, the board is symmetrical. So if there is a safe square in one half, there is a corresponding safe square in the other half. Secondly, the bishop placed by the second player is safe from the bishop that it is symmetric to. So it is safe from all the bishops already on the board.

Game 3

On a chessboard, a rook stands on the bottom left-hand square.

Players take turns moving the rook as many squares as they want, either horizontally to the right or vertically upward.

The player who can place the rook on the top right-hand square wins.

Working from the Endgame:

We will break the chessboard into Winning Positions (WPs) and Losing Positions (LPs). A WP is a square which, if you leave the rook there, you are guaranteed to win. A LP is a square where, if you leave the rook there, the other person can move to a WP.

Note: *If the game starts on a WP, the second person will win. This is because, from a WP, the first player must move to a LP. The second player now moves to a WP, and will win. If the game starts on a LP, the first player can win by moving to a WP.*

Step 1: *Initially, the only square that's definitely a WP is h8, the square we want to end up on. What we will do is put a W in this square.*

We now fill in all of the LPs emanating from this WP – that is, any square from which I can move the rook to the WP. We will colour these grey – they are the rest of row 8 and column h.

Step 2: We now want to find the next WP. This will be a square from which our opponent is **forced** to move into a LP. There is one such square: g7. From this square, the only possible moves are up to g8, or over to h7. So we put a W in here. Now we shade in all of the LPs that emanate from this WP, namely the rest of row 7 and column g.

								W

Step 1

								W
							W	

Step 2

We continue in this way, until, in Step 8, we have the whole board filled in:

								W
							W	
					W			
				W				
			W					
		W						
	W							
W								

The winning positions are on the main diagonal. So the second player wins, so long as they always move onto this diagonal. Note that, in their first move, the first player must move into a LP.

Game 4

On a chessboard, a king stands on the bottom left-hand square.

Players take turns moving the king either one square upwards, to the right, or along a diagonal going up and to the right.

The player who can place the king on the top right-hand square wins.

Work from the Endgame in order to determine the WPs, which are shown on the right. The first player can win, as long as they move into the first winning position on their first move.

	W		W		W		W
	W		W		W		W
	W		W		W		W
	W		W		W		W

Game 5

A queen stands on square b1. Players take turns moving the queen any number of squares upwards, to the right, or along a diagonal going up and to the right.

The player who can place the queen on h8 wins.

The winning positions are indicated below. The first player can win, as long as they move two squares to the right on their first go.

							W
					W		
						W	
		W					
W							
				W			
			W				

Game 6 (Nim)

A bundle of 32 matchsticks is put in the middle of the table.

Players take turns to remove one, two, or three sticks at a time. The winner is the person who removes the last stick.

Students will have figured out already that you want to be the person who leaves 28 matchsticks behind them – in that way, if the other person takes x matchsticks, you can take the remaining $4 - x$ matchsticks and win.

In our new way of thinking about it, if we write out the number of matchsticks left, we can work out the losing and winning positions (the losing positions are shaded in):

0	1	2	3	4	5	6	7	8	9	10	11	12	...	28	29	30	31	32
---	---	---	---	---	---	---	---	---	---	----	----	----	-----	----	----	----	----	----

0 left is a winning position; 1, 2, or 3 left are all losing positions. This means that 4 is a winning position; 5, 6, and 7 are losing positions. Working backwards, we can see that the winning positions are $4n$, for any $n \in \mathbb{N}$.

In general, if there are N matchsticks initially, and I can take away any number of matchsticks from 1 to c at a time, then the winning positions will be $(c + 1)n$, for any $n \in \mathbb{N}$. (Here, c is 3, so $(c + 1)n$ is $4n$.) So the second person will win if N is a multiple of $k + 1$, as in this example; the first person will win if it's not.

Game 7

A checker is placed at each end of a strip consisting of 1×20 squares.

Players take turns moving either checker in the direction of the other, by 1 or 2 squares. Neither checker can jump the other. The player who cannot move loses.

This is just a different version of Nim. As each checker takes up 1 square, there are 18 squares left. You can think of these as matchsticks, and each move as removing 1 or 2 matchsticks. So the second player will win, because the winning positions are 0, 3, 6, 9, 12, 15, and 18 squares left to traverse. If the first player moves a checker x squares, the second player moves either checker $3 - x$ squares.

What would happen if it was a strip of 1×21 squares?

In this case, there are 19 squares left to traverse, so the first player should move 1 square, so that there are 18 left. The second player now adopts the winning strategy from the first part of the question in order to win.

Week 18: Gargantuan Games

Introduction

This is a Whole-Class lesson. It is the third and final lesson involving games. It uses more complex versions of *Working from the Endgame* introduced in the last lesson, as well as some symmetric strategies.

Resources

- Square paper for Games 1 and 2.
- Coins or counters for the rest of the games.
- A copy of the Gargantuan Games Activity Sheet, one per student.

Activities

Whole-Class Lesson:

Once again, students are split into pairs (or groups, with each group divided into 2 teams).

Students initially play Games 1 and 2 themselves – if they are having trouble, the teacher should point out that they might try symmetric strategies, like last week.

Students then try Games 3, 4, and 5 – if they are having trouble here, the teacher might suggest *Working from the Endgame*.

Students are unlikely to finish all of the games on the sheet. This being the last week of this handbook, it's probably as well to leave them with something to think about!

Take-home Problem:

Finish off games on Gargantuan Games Activity Sheet.

Gargantuan Games

In each case, figure out who wins, and how!

Game 1

Two players take turns placing **x**'s and **o**'s on a 7×7 chessboard. The first player places **x**'s, and the second player places **o**'s. Once all the squares have been filled in, the first player gets a point for each row or column that contains more **x**'s than **o**'s. The second player gets a point for each row or column that contains more **o**'s than **x**'s. The player with the most points wins.

Game 2

Two players take turns breaking a piece of chocolate consisting of 5×10 small squares.

At each turn, they may only break along the division lines of the squares.

The player who first obtains a single square of chocolate wins.

Game 3

There are 50 coins in a pile.

At each turn, a player removes no more than half of the coins in the pile.

The player who cannot take a turn loses.

Game 4

There are two piles of coins, one with 20 coins and the other with 21 coins.

At each turn, a player discards all of the coins in one pile, and splits the remaining pile in two.

The player who cannot take a turn loses.

Game 5

The number 60 is written on a blackboard.

Players take turns subtracting from the number on the blackboard any of its divisors, and replacing the original number with the result.

The player who writes the number 0 loses.

E.g. The first player could subtract 1, 2, 3, 4, 5, 6, 10, 12, 15, or 30 from 60. If they subtracted 60 from 60, they'd get 0, and so would lose.

Solutions

Game 1

Two players take turns placing **x**'s and **o**'s on a 7×7 chessboard. The first player places **x**'s, and the second player places **o**'s. Once all the squares have been filled in, the first player gets a point for each row or column that contains more **x**'s than **o**'s. The second player gets a point for each row or column that contains more **o**'s than **x**'s. The player with the most points wins.

*This game has a symmetric strategy. The first person puts an **x** in the centre square. For each **o** the second player places, the first player responds by placing an **x** in the symmetric square, under central symmetry in the centre square. By symmetry, the first player will then gain a point for the centre row and centre column. For any other row that the second player wins, the first player will win the row that is its image under central symmetry in the centre square. As the case is similar with columns, the first player will win by 2 points, i.e. 8 points to 6.*

Game 2

Two players take turns breaking a piece of chocolate consisting of 5×10 small squares.

At each turn, they may only break along the division lines of the squares.

The player who first obtains a single square of chocolate wins.

The loser is the person who breaks off a row of chocolate that is $n \times 1$. The first player can win by breaking the bar into two 5×5 squares, and playing symmetrically from there.

Game 3

There are 50 coins in a pile.

At each turn, a player removes no more than half of the coins in the pile.

The player who cannot take a turn loses.

The winning positions are those in which there are $2^n - 1$ coins. So the first person takes 19 coins to leave 31, and will win from here.

To see this, we'll work backwards from the final winning position (WP), which is to leave 1 coin, so that the other player cannot make a move:

WP1	Leave 1 coin	Other player can't remove any coins, so you win.
LP1⁴	Leave 2 coins	You can remove 1 coin, to achieve WP1.
WP2	Leave 3 coins	The other player can only remove 1 coin (at most

⁴ Losing Position 1, the first Losing Position.

		<i>half the pile), so they must go to LP1.</i>
LP2	<i>Leave 4, 5, or 6 coins</i>	<i>If there are this number of coins left, you can remove 1, 2, or 3 coins respectively to go to WP2.</i>
WP3	<i>Leave 7 coins</i>	<i>The other player must remove 1, 2, or 3 coins. This causes them to enter LP2.</i>
LP3	<i>Leave 8, 9, 10, 11, 12, 13, or 14 coins</i>	<i>From any of these positions, you can remove exactly enough coins to leave 7 coins, and so be in WP3.</i>
WP4	<i>Leave 15 coins</i>	<i>The other player must remove between 1 and 7 coins, inclusive. Any such move puts them in LP3.</i>

Continuing on in this way, it can be seen that the WPs are to leave 1, 3, 7, 15, or 31 coins. To get from one WP to the next, double it and add 1. Alternatively, these are all of the form $2^n - 1$, as mentioned above.

Game 4

There are two piles of coins, one with 20 coins and the other with 21 coins.

At each turn, a player discards all of the coins in one pile, and splits the remaining pile in two.

The player who cannot take a turn loses.

The first player discards the pile with 21 coins, and splits the one with 20 coins into two piles, each with an odd number of coins. At each turn now, the second player will have to discard one of the odd-numbered piles, and split the other pile into two – one odd- and one even-numbered. The first player then discards the odd-numbered pile, and splits the even-numbered pile into two odd-numbered ones. This guarantees victory.

To see why this will always work, work backwards from the final WP, which is to leave two piles, each with 1 coin:

WP1	<i>1 and 1</i>	<i>The other player can divide neither bundle in 2, so they lose.</i>
LP1	<i>2 and 1, 2 and 2, 2 and 3, 2 and 4, etc.</i>	<i>I discard the second pile, and split the first pile into 1 and 1, thus entering WP1.</i>
WP2	<i>3 and 1, 3 and 3</i>	<i>No matter which pile the other player discards, they will be left with 3 coins. These must be split into 2 and 1, entering LP1.</i>

LP2	4 and 1, 4 and 2, 4 and 3, 4 and 4, etc.	<i>I can discard the second pile, and split the first pile into 3 and 1. Alternatively, if you leave 4 and 6, I can discard the 4 and leave 3 and 3.</i>
WP3	5 and 1, 5 and 3, 5 and 5	<i>If the other player leaves 5 coins in the pile they don't discard, they must enter LP1 or LP2. If they leave 3 coins in the pile they don't discard, they must enter LP1. Either way, they lose.</i>

Continuing on in this way, if I leave 2 piles with an odd number of coins after I've taken my turn, I will win.

Game 4

The number 60 is written on a blackboard.

Players take turns subtracting from the number on the blackboard any of its divisors, and replacing the original number with the result.

The player who writes the number 0 loses.

The WPs are the odd numbers. Thus the first player can win as long as they start by subtracting an odd factor.

To see this, we'll work backwards from the final WP, which is to write the number 1 on the board:

WP1	1	<i>The other player must now subtract 1, and so loses.</i>
LP1	2	<i>This is the only number that has a factor that's 1 less than the number itself.</i>
WP2	3	<i>The other player must subtract 1, which puts them in LP1.</i>
LP2	4, 6	<i>You can subtract 1 or 3, respectively, to get to WP2.</i>
WP3	5, 7	<i>Either of these leads to LP2.</i>
LP3	8, 10, 14	<i>You can get to WP3 from any of these.</i>
WP4	9, 11	<i>9 must lead to 8 or 6; 11 must lead to 10.</i>
LP4	12, 18, 22	<i>Each of these can lead to WP4.</i>

As is indicated from the table, it turns out that the WPs are the odd numbers, and the LPs are the even numbers.

A Short History of Maths Circles

The origins of the Maths Circles can be traced back as early as a century ago in Eastern Europe, where they still flourish. They function at many levels: from in-school to city-wide circles, to national training camps. Printed resources for these activities include the popular Kvant Magazine in Russia and the 117 year old Gazeta Matematica in Romania. They contain articles by teachers and professional mathematicians as well as problems and solutions by teachers and students alike. Communication between academics, teachers and students is typical for Eastern European circles.

As a social consequence, mathematics as a school discipline enjoys great prestige and respect in these countries. Today, it is not that surprising to find a successful writer publicly reminiscing about his school-boy elation in solving a challenging maths problem, or the Romanian president himself reported as an ex-Gazeta Matematica problem-solver. Members of the general public receive news of the annual results of the national teams at international problem-solving competitions with joy and national pride reminiscent of the regard enjoyed by Olympic athletes in countries worldwide.

But the most important social aspect of maths circles is the cultivation of an ethos of giving. Indeed, most maths circle participants preserve a lifelong gratitude for the generous efforts of teachers, mathematicians, university students who enlightened and delighted them in the after-school hours. Given an opportunity, they pay back this debt to society by mentoring new generations. This creates a virtuous cycle in mathematical education.

The full social significance of this phenomenon can only be perceived in a global context. After 1989, when many Eastern Europeans chose the relative personal freedom and comfort of the West, this was a testimony to the political, moral and economic failure of communism. But the fact that so many of them integrated in their host countries as successful mathematicians, scientists, engineers is a testimony to the benefits bestowed on these people by their education. In the 1990's, when the idea of maths circles captured attention in the US, it was enthusiastically supported by many of these émigré academics. The maths circles spread like wild-fire through American Universities, everywhere finding mathematicians delighted to share the elegant and imaginative word of mathematics with ever younger people. Invaluable electronic and printed resources were developed. Research Institutes

brought together interested parties in workshops and conferences. The crucial role played by universities and research institutes in their development and day-to-day running is a particular feature of the US Maths Circles. Another specific aspect is the diversity in the age of participants, with students from 10 to 18 years of age sometimes participating in the same circle. Indeed, an underlying current was noticed which seems to drive the starting ages of audience members ever lower... There are various types of mathematics circles/enrichment type activities in many other countries: Australia, Canada, Germany, Israel, New Zealand, Saudi Arabia, South Africa to name but a few. Many of these activities receive some support from the state or from local universities. Many revolve around a calendar of maths competitions starting from the international Kangaroo competition and culminating in the International Mathematical Olympiad (IMO). They all have in common the sense of wonder and elation experienced by participants:

“... we all felt that nothing out there could even compare to what we were doing. That was when I decided that I would major in math in college. As I studied more mathematics over the next ten years, the problems got harder, the lectures got more complicated, but the feeling that there is nothing better I could possibly do with my time is still there.”

***Maria Chudnovsky, professor, Columbia University;
from an interview at Clay Mathematics Institute***

It should also be mentioned that from the outset, maths circles became part of a diverse group of circles on topics ranging from creative writing to physics and computer science. Just like young athletes are supported through sports teams and opportunities to participate in competitions, budding musicians through practice at music clubs, concerts and competitions, it is natural to support and encourage various kinds of in-depth intellectual exploration from an early age.

Maths Circles in Ireland

Mathematical enrichment programmes for secondary school students have been developed since 1987 by five Irish centres: UCC, UCD, University of Limerick, NUI Galway, NUI Maynooth. With cultivating a love and appreciation for mathematics as primary aim, these programmes also prepare a small group of students for competing internationally in problem solving. While many participants have acknowledged benefits from these programmes, there has also been recognition among the organisers that in terms of scope and audience reached, this may be too

little, too late – and sometimes also, ironically, trying to do too much too fast, in an attempt to catch up with international standards.

"We could have a broader basis for Irish success at the IMO if more students gained experience in creative mathematical problem solving towards the end of their Primary School age. We would be happy to support teachers with our experience..."

Dr. Bernd Kreussler, lecturer and math enrichment organiser, MIC, UL

On the other hand, Maths school clubs led by dedicated teachers have existed in some Irish schools for quite a while. However, building these clubs, preparing activities, selecting resources, searching for suitable mathematical content all on top of the usual working hours is a hugely consuming task to be done in isolation. For this reason, a network of support for organisers seemed to be a first necessity for these activities.

In October 2012 the organiser of the Maths Day at UCC, Edwin O'Shea (now of James Madison University), offered me an opportunity to give a short talk about these issues. It was in this context that I was lucky to meet two recent UCC PhD graduates, David Goulding and Julie O'Donovan, who got excited about the project of building a network of Maths circles in Ireland, united by the same ethos/purpose and supporting each other through shared resources. Our feelings in regards to chances of success ranged from determined to cautiously optimistic to sceptical, but we all agreed that chances can be greatly enhanced through communication among the various members of the mathematical community: teachers as well as professors and lecturers, college students and postgraduates, even students' parents. We all benefit from the health of the mathematical education in this country...

After evenings of careful planning and negotiations, the first maths circle was held in *North Presentation School* in Cork run by Julie, David, and UCC students Robert Linehan and Patrick Gorman, in a playful and experimental style. The feedback collected at the end of the four weeks encouraged us to continue with the next step of the project.

Next we engaged the help of UCC Mathematical Sciences lecturers, postdocs and especially students, who launched enthusiastically in preparing a workshop for teachers and other would-be maths circle organisers. The workshop was held in March 2011 and followed by another in October 2011. The attending teachers found

they liked our UCC students so much that they invited them to help with setting up and running maths circles in their schools.

It was our great stroke of luck that Ciarán Ó Conaill from Douglas Community School, himself a UCC graduate, was among the first workshop participants. In Autumn 2011, he became the first teacher to organize a maths circle in his school. JP McCarthy, a UCC postgraduate student, helped run the first four classes, after which the circle ran independently for much of the duration of the school year. It is from this experience that Ciarán has drawn in putting together the lesson plans in this handbook.

Other strokes of luck followed soon: a generous NAIRTL grant; the hugely helpful IMTA Cork branch who granted us a stand at Cork Maths Fest 2011 and 2012; the more than 30 UCC students who registered to help with the project; and last but not least, the teachers and students who ran the circles:

- *Douglas Community School*: Ciarán Ó Conaill (Teacher) and JP McCarthy (UCC Student).
- *St. Aidan's Community College*: Joan Hough (Teacher) and Grainne Walsh (UCC Student).
- *Colaiste An Phiarsigh*: Sian Joyce (Teacher) and JP McCarthy (UCC Student).
- *Scoil Mhuire Gan Smál*: Sean Foley (Teacher) , Sinead McCarthy and Joe McEniry (UCC Students).
- *Nagle Community College*: Ciara Twomey and Kathleen Cronin (Teachers), Meabh Kennedy and Sinead O'Sullivan (UCC Students).
- *Hamilton High School*: Eoghan O'Leary (Teacher) and Aodhan O'Leary (UCC Student).
- *Mount Mercy College*: Mary Cowhig and Ann Sullivan (Teachers), Sophie Daly and Sophie Scannell (UCC Students).
- *North Presentation Secondary School*: Julie O'Donovan, David Goulding (Maths Circles Team), Robert Linehan and Patrick Gorman (UCC Students).
- *Bishopstown Community School*: Mary Sheehan (Teacher), Vahid Yazdanpanah and Phillip O'Mahony (UCC Students).
- *Gaelcholáiste Mhuire*: Valerie Mulcahy (Teacher) and Eoin O'Mahony (UCC Student).
- *St. Aloysius*: Sarah Moore (Teacher) and Sorcha Gilroy (UCC Student).
- *St. Mary's Secondary School*: Sharon O'Connell (Teacher) and Deirdre Van Der Krogt (Maths Circles Volunteer).

“When I first began giving the maths circles the students were so quiet and almost afraid to guess and answer or voice an opinion but last week they amazed both me and Joe. On hearing a problem they automatically started to think of a method of solution not just a guess at the answer. Once they had found a way of getting a solution they voiced their opinion without any fear of being incorrect. Over the past few weeks they have learned that most of maths is trial and error and that you will almost never get anything right first time. It was great to see how much confidence they had gained in their own abilities and the way they approached problems with certainty.”

Sinead McCarthy, UCC Student

What's next?

As the Maths Circles are starting their second year in Ireland, we hope that:

- Teachers around Ireland will feel compelled to give this handbook a tryout and start their own maths circles for beginning Junior Cycle students. Some may find that they need to adapt the material to the particular needs and tastes of their students, as well as their particular schedules. They should feel free to select the lesson plans as they feel most appropriate.
- We'll successfully start off our follow-up programme for those 2nd and 3rd year students who have already attended maths circles in their first year, as well as students selected by other schools. In 2012-2013, this will run as a Junior Maths Enrichment programme at UCC. We are greatly indebted to UCC student Kieran Cooney for taking so many hours of his summer holidays to prepare lesson plans for these 2nd and 3rd year students, thus making possible this follow-up programme.
- University students will be able to continue helping out with running maths circles around their home university. Their energy and enthusiasm for the project are invaluable.
- Parents of maths circle participants will talk with their children about their experience. If they find their children enthusiastic, we hope they'll spread the word, and help us find sponsors.
- Companies and public agencies will recognise the potential and become sponsors. With relatively few resources, we can reach many children.
- The maths circle community will keep in contact with us through contact@mathscircles.ie and even send in their favourite lesson plans for publication on our webpage: www.mathscircles.ie.

Anca Mustata

Select Bibliography

With a book such as this, it is not possible to give a complete list of sources – many of the puzzles have appeared in a number of guises down through the years. So the following is an attempt to try and apportion credit, without being in any way definitive or exhaustive:

- A number of the lesson plans here were adapted, to a greater or lesser extent, from *Mathematical Circles: Russian Experience (Mathematical World, Vol. 7)*, by Fomin, Genkin, and Itenberg, which is a fantastic resource.
- The 7 S logical experiment in the Bothersome Brainteasers in Week 3 is taken from Steven Pinker's *How The Mind Works* (which also has some nice observations about probabilities and common sense).
- The Leaping Lizards activity from Week 7 was inspired by a game from a US company called FunThink.
- The 1089 trick can be found in many places – I saw it first in David Acheson's *1089 and All That*, while www.cut-the-knot.org is another place where it appears that's well worth a look.
- I first saw the 9-card trick on YouTube at: <http://www.youtube.com/watch?v=POQTbs2YfFs>. The person doing the trick in this video has no idea how it works.
- The Christmas Tree counting question was prepared for a workshop by Conor Leonard, a student at UCC.