MATHEMATICS ENRICHMENT - POLYNOMIALS 2

1. Find all solutions to the following equations:
a)$x^3 - 5x^2 + 4 = 0$.

Hint: If $x = a$ is a solution of the above equation, then $a^3 - 5a^2 + 4 = 0$ so $4 = -a^3 + 5a^2 = a^2(-a+5)$. Thus if a is an integer, then it must be a divisor of 4.

b)$x^3 + 2x^2 - x + 6$.

Hint: Integer solutions must be divisors of 6.

c)$x^3 - 6x^2 + 12x - 10 = 0$.

Hint: First complete the cube $x^3 - 6x^2 + 12x + 8 = (x-2)^3$.

2. Find the sums $S_n = r_1^n + r_2^n + r_3^n$ for $n = 1, 2, ...5$ if r_1, r_2 and r_3 are the roots of the equation

$$x^3 - 2x^2 + 2x + 3 = 0.$$

Hint: Writing $x^3 - 2x^2 + 2x + 3 = (x - r_1)(x - r_2)(x - r_3)$ and multiplying the right hand side we get $r_1 + r_2 + r_3 = 2$ as well as $r_1r_2 + r_1r_3 + r_2r_3 = 2$ and $r_1r_2r_3 = -3$.
Then check $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_1r_3 + r_2r_3)$ and use this to find S_2.
To find S_3, note that substituting $x = r_i$ in the equation

$$\begin{aligned}
x^3 - 2x^2 + 2x + 3 &= (x - r_1)(x - r_2)(x - r_3) \\
\Longrightarrow r_1^3 - 2r_1^2 + 2r_1 + 3 &= 0. \\
r_2^3 - 2r_2^2 + 2r_2 + 3 &= 0. \\
r_3^3 - 2r_3^2 + 2r_3 + 3 &= 0.
\end{aligned}$$

Add the last three equations to find $S_3 - 2S_2 + 2S_1 + 9 = 0$ and hence find S_3.
To find S_n, first multiply the equations above by the suitable r_i^{n-3} and then add them up to find $S_n - 2S_{n-1} + 2S_{n-2} + 3S_{n-3} = 0$.

3. Find the numbers a and b knowing that if you subtract the number 7 from each of the roots of the equation $x^2 + ax + b = 0$, you will obtain the roots of $x^2 + bx + a = 0$.

Hint: Write $x^2 + ax + b = (x - r_1)(x - r_2)$, multiply through and compare to $(x - r_1 + 7)(x - r_2 + 7) = x^2 + bx + a$. 4. A certain polynomial $p(x)$ has the values $p(2) = 4$ and $p(-1) = 5$. If you divide $p(x)$ by $x^2 - x - 2$, what remainder will you get? (First decide what will be the degree of the remainder).

Hint: Suppose that long division of $p(x)$ by $x^2 - x - 2$ gives us $q(x)$ together with a remainder $r(x)$. Note that we can keep dividing by $x^2 - x - 2$ until we get $r(x)$ of degree 1, namely of the form $r(x) = Ax + B$.

$$p(x) = (x^2 - x - 2)q(x) + Ax + B.$$

Substitute $x = 2$ and $x = -1$.

5. Let $p(x)$ be a polynomial of degree ≤ 2 and a, b, c be some numbers. Prove that

$$p(a+b+c) - p(a+b) - p(a+c) - p(b+c) + p(a) + p(b) + p(c) - p(0) = 0$$

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First check that the equation holds for any $p(x) = x^n$.