MATHEMATICS ENRICHMENT - POLYNOMIALS

Q1. Find all integers n for which the following expressions are perfect squares:

a) $n^2 + 20n + 11$
b) $n^2 + 20n + 12$
c) $n^2 - 11n + 63$

Hint: Complete the square $(n + y)^2$ where $y =$ half the coefficient of n. Then use the difference of squares formula.

Q2. Show that for any integer $n > 11$, the expression $n^2 - 19n + 89$ is never a perfect square.

Hint: Try the same method as in Q1, prove that your solutions are not integers.

Q3. Show that for any non-negative integer n, that $3^n + 2(17^n)$ is never a perfect square.

Note: This problem is not really about polynomials so its solution will come by different methods. For example, taking $n = 1, 2, 3, 4, ...$ you can systematically check the last digit of $3^n + 2(17^n)$. Can a perfect square end in those digits?

Q4. Simplify the following

a) $(a - b)^3 + (b - c)^3 + (c - a)^3$
b) $(a + 2b - 3c)^3 + (b + 2c - 3a)^3 + (c + 2a - 3b)^3$

Hint: a) Method I: Use $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$ for the sum $(a - b)^3 + (b - c)^3$. After simplifying, can you find a common factor for all 3 terms in the sum at a)? Method II: Think of this as a polynomial in the variable a like this:

$$P(a) = (a - b)^3 + (b - c)^3 + (c - a)^3.$$

Think of b and c as numbers. Sub in values for a to find how to factor the polynomial.
b) Method I: Use a) where a, b, c are replaced by $a - 2c$, $c - 2b$, $b - 2a$. Method II Same idea as in Method II above.

Q5. Show that $(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(b + c)(c + a)$.

Hint: Method I: Multiply through. Method II: Think of these as polynomials in the variable a and find its factors. Also use the symmetry in a, b, c.

Q6. Factorise the following expressions

a) $x^4 + x^2 + 1$.
  a) Hint: Add and subtract x^2 and use differences of squares.
b) $x^5 + x + 1$
  b) Hint: Add and subtract x^2 and use $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

c) $x^9 + x^4 - x - 1$

c) Hint: Group the 4 terms of the sum in pairs of 2 and use various factorization formulae like for example

$$x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) \text{ or}$$

$$x^8 - 1 = (x^4-1)(x^4+1) = (x^2-1)(x^2+1)(x^4+1) = (x-1)(x+1)(x^2+1)(x^4+1)$$

d) $x^{12} - x^4 - x^3 + 1$

d) Hint: Similar to the above

e) $x^{10} + x^5 + 1$

e) Hint: What is the remainder when dividing by $x^3 - 1$?

Q7. Find all positive integers n for which $n^8 + n + 1$ is prime.

Hint: First, look at this as a polynomials and prove $n^2 + n + 1$ is a factor. Find the other factor as well. Now use the fact that this should be a prime number.

Q8. Prove or disprove the following statement: For every positive integer n, the greatest common divisor of $5n + 4$ and $9n - 7$ is 1.

Hint: Use Euclid's algorithm.

Q9. Show that if x is a non-zero real number, then $x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0$.

Hint: Multiply by x^4 and factorize.

Q10. Let $n > 0$ be an integer. Find the remainder upon division of $x^n + x^{n-1} + ... + 1$ by (a) $x^2 + 1$ and (b) $x^2 + x + 1$.

Hint: Just do the long division carefully.

Q11. Find the greatest common divisor of the following pairs of polynomials

a) $f(x) = x^2 - 1, g(x) = x^3 - 1$
b) $f(x) = x^2 + x, g(x) = x^3 - x^2 + x$
c) $f(x) = x^3 - 3x - 2, g(x) = x^3 + x^2 - 3x - 6$
d) $f(x) = x^3 + 3x^2 + 5x + 3, g(x) = x^4 + x^3 + x^2 + 2x + 1$
e) $f(x) = x^4 - 16, g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16$
f) $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1, g(x) = x^3 + x^2 + x + 1$
g) $f(x) = x^6 + x^4 + 2x + 1, g(x) = x^7 - x^5 + x^4 + 3x^3 - 2x^2 + 2$

Hint: With some of these, you can try factoring out both polynomials and picking the common factors. With others, you may try Euclid's algorithm, just like you would do for numbers.

Q12. For the following find the greatest common divisor and write it as a linear combination of these two polynomials, namely write $d(x) = a(x)f(x) + b(x)g(x)$ for some $a(x)$ and $b(x)$ polynomials.

a) $f(x) = x^3 - 3x - 2, g(x) = x^3 + x^2 - 3x - 6$
b) $f(x) = x^3 - 1, g(x) = x^4 - x^3 - x^2 + 1$

c) $f(x) = x^3 + 1, g(x) = x^5 + 1$

Hint: Euclid's algorithm, just like you would do for numbers.

Q13. Let $n > 1$ be an integer. Prove that $n^5 + n + 1$ is divisible by at least two distinct primes.

Hint: Factorize $n^5 + n + 1$ by a trick used in similar problems above. Then try to find g.c.d. of the factors.

Q14. Show that if n divides m, then $x^n - 1$ divides $x^m - 1$.

Hint: Say $m = nk$. Substitute $y = x^n$ above.

Q15. By testing a few values, guess the value of $\gcd(x^n - 1; x^m - 1)$ for arbitrary $n; m > 0$.

Hint: Use Q14.

Q16. Find all positive integers n such that $x^{2n+1} + x + 1$ is divisible by $x^n + x + 1$.

Hint: Do the long division

Q17. Given that a polynomial $f(x)$ divides another polynomial$g(x)$, show then that $f(x)$ also divides the sum of these polynomials, $f(x) + g(x)$, and the difference of the polynomials, $g(x) - f(x)$. With this fact in mind, try approach $Q16$ in a different manner.

Hint: Try applying Euclid's algorithm.