UCC Maths Enrichment - Number Theory

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Exercises

1. Recall that given two integers a and b we say that $a \mid b$ if there exists another integer c such that $ac = b$. Prove the following:
   - If $a \mid b$ and $a \mid c$ then $a \mid b + c$
   - If $a \mid b$ and $b \mid c$ then $a \mid c$
   - If $a \mid b$ and $a \mid c$ then $a \mid bc$
2. Show that no two consecutive integers can have a common prime factor.
3. Find $gcd\,(98765, 4321)$
4. Given two positive integers a and b, we define their lowest common multiple to be the smallest positive integer such that it is divisble by both a and b. We denote this number by $lcm\,(a, b)$. Prove that $ab = gcd\,(a, b)\, lcm\,(a, b)$.
5. How many positive divisors does 2015 have?
6. Find all positive integer solutions to $a^3 b = 5,000,000$
7. Find the remainder of 7^{7^7}upon division by 9.
8. Find all integer solutions to $a^4 + b^4 + c^4 = 99^{99}$
9. Find the inverse of 75 $(mod\,101)$.
10. Construct a table of the first 50 values of the Euler totient function, $\varphi\,(n)$.

Problems

1. Find all right triangles with integer side lengths such that the area and perimeter are equal.
2. Prove Fermat's Last Theorem for the case $n = 4$, i.e. show that
$$x^4 + y^4 = z^4$$
has no integer solutions.
3. *Prove or disprove*: there exists 100 consecutive integers, none of which are prime.
4. Given positive integers x and y, with $x < y$, find all the solutions to
$$x^y = y^x$$
5. Find all integer solutions to $a^5 + 3b^5 = 72$.
6. For p a prime, solve for the integers x and y
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$$
7. Find all positive integers that have an odd number of divisors.
8. Prove that the fraction $\frac{21n+4}{14n+3}$is irreducible for every natural number n.
9. Show that there are infinitely many prime numbers of the form $4k + 3$, with k and integer.
10. Find all positive integers n such that $\varphi\,(n)$, the Euler totient of n, is prime valued.