Inequalities

- Arithmetic Mean-Geometric Mean (AM-GM) Inequality
  If $x_1, x_2, \ldots x_n \geq 0$, then:
  $$\frac{x_1 + x_2 + \ldots x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$
- Cauchy-Schwartz (CS) Inequality:
  If $x_1, x_2, \ldots x_n \in \mathbb{R}$ and $y_1, y_2, \ldots y_n \in \mathbb{R}$ then:
  $$(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \ldots x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2)$$
- Rearrangement Inequality: If $x_1 \leq x_2 \leq \ldots \leq x_n$, $y_1 \leq y_2 \leq \ldots \leq y_n$ and $z_1, z_2, \ldots z_n$ is a permutation of $x_1, x_2, \ldots x_n$ then:
  $$x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \geq z_1 y_1 + \ldots + z_n y_n \geq x_n y_1 + x_{n-1} y_2 + \ldots + x_1 y_n$$
- Jensen's Inequality: If $f(x)$ is a convex function on the domain $[a, b]$, $a_1, a_2, \ldots a_n \in [0, 1]$ with $a_1 + a_2 + \ldots + a_n = 1$, and $x_1, x_2, \ldots, x_n \in [a, b]$ then:
  $$a_1 f(x_1) + a_2 f(x_2) + \ldots a_n f(x_n) \geq f(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)$$
  If $f(x)$ is a concave function, then the inequality is reversed.

Warm-Up Problems

1. Use the AM-GM Inequality to maximise the following functions $\forall x \in [0, 1]$:
   (a) $x(1 - x)$
   (b) $x^3(1 - x)$
   (c) $x(1 - x^2)$
   (d) $x^4(1 - x^5)$
2. Use the AM-GM Inequality to prove the following $\forall x, y, z \in \mathbb{R}^+$:
   (a) $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9$
   (b) $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ (or use Jensen's Inequality)
3. If x and y are positive, use the Rearrangement Inequality to show that:
   $$\sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} \geq \sqrt{x} + \sqrt{y}$$
4. If a, b and c are positive, show that:
   $$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$$
5. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Use the Cauchy Schwartz Inequality to show that:
   $$\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n} \leq \sqrt{n}$$

Exercises

For the following problems, choose carefully which inequality to use.

1. $\forall x, y, z \in \mathbb{R}$, Show that:
$$x^2 + y^2 + z^2 \geq |xy + xz + yz|$$
2. Find the maximum volume of a cube with surface area S (with respect to S).
3. Let a, b, x and y be positive real numbers. Show that:
$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$$
4. Let a and b be positive numbers such that $a + b = 1$ and $a, b \neq 0$. Show that:
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$$
5. Let $a'_1, a'_2, \ldots, a'_n$ be a permutation of $a_1, a_2, \ldots, a_n$ which are positive real numbers. Show that:
   (a) $a_1^2 + a_2^2 + \ldots + a_n^2 \geq a_1 a'_1 + a_2 a'_2 + \ldots + a_n a'_n$
   (b) $\frac{a'_1}{a_1} + \frac{a'_2}{a_2} + \ldots + \frac{a'_n}{a_n} \geq 3$
6. Let x, y and z be positive real numbers. Show that:
$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geq \frac{9}{x+y+z}$$

Homework

If you have time, attempt to write solutions for the following problems. If you hand them up to me next time you see me, I'll correct them for you.

1. For $a, b, c \in \mathbb{R}^+$, show that:
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{b+a} \geq \frac{3}{2}$$
2. For any n positive real numbers $x_1, x_2, \ldots x_n$, such that $x_1 + x_2 + \ldots + x_n = 1$, show that:
$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \ldots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{x_1 + x_2 + \ldots + x_n}{\sqrt{n-1}}$$
   (Hint: $f(x) = \frac{x}{\sqrt{1-x}}$ is convex on $(0, 1)$.)
3. Let a, b and c be the lengths of the sides of a triangle. Show that:
$$a^2 b(a-b) + b^2 c(b-c) + c^2 a(c-a) \geq 0$$