UCC Maths Enrichment - Methods of Proof

Tutor: Kieran Cooney **Due:** 14-2-15

Exercises

These problems are designed for you to practice the techniques we developed in class; induction, contradiction, the pigeonhole principle and the extreme principle.

1. Prove that for all natural numbers n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

2. Prove that for all natural numbers n (in two ways if you can), .

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

3. Prove that if n is an integer greater than 3, then $n! > 2^n$.

4. Is $\sqrt{2} + \sqrt{3}$ a rational number?

5. Prove that $\sqrt[n]{p}$ is an irrational number, for p a prime, n an integer greater than 2.

6. Prove that if a is a rational number and b is an irrational number, then $a + b$ is an irrational number.

7. Prove that of any 10 points chosen within an equilateral triangle of unit side length, there are two whose distance apart is at most $\frac{1}{3}$.

8. Thirteen schools took part in an athetlics competition. There were 1514 student spectators. Show that there was a school that was cheered on by at least 117 students.

9. There are 2,000 points on a circle, and each point is given a number which is equal to the average of its two nearest neighbours. Show that all the numbers must be equal.

Problems

Please attempt questions 2, 3, 6 and 9 for the due date.

1. Three lines are said to be in *general position* if they all do not meet at the same point and no pair of the lines are parallel. Into how many regions is the plane divided by n lines, any 3 of which are in general position?

2. Let $p_n$denote the n-th prime number, so that $p_1 = 2$ and $p_{11} = 31$. Prove that $p_n \leq 2^{2^n}$.

3. Again, find the least number of moves for n disks in order to complete the Tower of Hanoi puzzle, except this time disks can only move one peg at a time. So a disk can move from pegs 1 to 2, and from pegs 2 to 3, but not from pegs 1 to 3.

4. Is it possible to place 1995 different natural numbers along a circle so that for any two of these numbers, the ratio of the greatest to the least is a prime?

5. Let $M(n)$ be the smallest integer such that if we have that many points in an equilateral triangle of unit side length, then we know that there must be two of them whose distance apart is $\frac{1}{n}$ units. Compute $M(n)$.

6. Inside a cube of side 15 units there are 11,000 given points. Prove that there is a sphere of unit radius within which there are at least 6 of the given points.

7. Colour the plane in 2 colours. Prove that one of these colors contains pairs of points at *every* mutual distance.

8. Let B and W be finite sets of black and white points respectively in the plane, with the proeprty that every line segment which joins two points of the same colour contains a point of the other colour. Prove that both sets must lie on a single line segment.

9. On a large, flat field n people ($n > 1$) are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry.