Polynomials

1. Factorizations

Warm-up: $x^2-y^2=(x-y)(x+y)$ and $x^2+2xy+y^2=(x+y)^2$. Application:

Completing the square to write a quadratic in vertex form:

Writing $ax^2+bx+c=a(x-h)^2+k$ for some numbers h,k.

Example:

$3x^2+30x+48=3(x^2+10x+16)=3(x^2+2\cdot5\cdot x+25-9)=3[(x+5)^2-9]$.

Why is this useful?

a) Solving quadratic equations: $3x^2+30x+48=0$.

Solution:

$3x^2+30x+48=0$ becomes $3[(x+5)^2-9]=0$. Using the difference of two squares this becomes: $3(x+5-3)(x+5+3)=0$ so $3(x+2)(x+8)=0$ so $x=-2$ or $x=-8$.

b) Find all integers n such that $n^2+6n+16$ is the square of an integer number.

Solution: We are asked to find integer solution for $n^2+6n+16=m^2$

But $n^2+6n+16=n^2+2\cdot3\cdot n+9+7=(n+3)^2+7=m^2$ becomes $(n+3)^2-m^2=-7$. Using the difference of squares: $(n+3+m)(n+3-m)=-7\cdot1=7\cdot(-1)$.

We solve simultaneous equations: $n+3+m=-7$ and $n+3-m=1$. Then $2n+6=-6$ so $n=-6$.

Or $n+3+m=7$ and $n+3-m=-1$ so $2n+6=6$ so $n=0$.

More formulae: Check the formulae below by multiplying through the right-hand-side, using distributivity:

$\boxed{x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+...+xy^{n-2}+y^{n-1})}$ for positive integers n;

$\boxed{x^n+y^n=(x-y)(x^{n-1}-x^{n-2}y+...-xy^{n-2}+y^{n-1})}$ for n **odd**. What can you say about the right-hand-side when n is even?

For example:

- $x^2-y^2=(x-y)(x+y)$;
- $x^3-y^3=(x-y)(x^2+xy+y^2)$;
- $x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)$;
- $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$; and

- $x^3+y^3=(x+y)(x^2-xy+y^2)$;
- $x^5+y^5=(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$.

$\boxed{x^n+C_1^n x^{n-1}y+C_2^n x^{n-2}y^2+...+C_{n-2}^n x^2y^{n-2}+C_{n-1}^n xy^{n-1}+y^n=(x+y)^n}$.

For example:

- $x^2+2xy+y^2=(x+y)^2$;
- $x^3+3x^2y+3xy^2+y^3=(x+y)^3$;
- $x^4+4x^3y+6x^2y^2+4xy^3+y^4=(x+y)^4$;
- $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5=(x+y)^5$;
- $x^2-2xy+y^2=(x-y)^2$;
- $x^3-3x^2y+3xy^2-y^3=(x-y)^3$;
- $x^4-4x^3y+6x^2y^2-4xy^3+y^4=(x-y)^4$;
- $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5=(x-y)^5$;

Example: Factoring polynomials by combining factorization formulae:

- $x^4-y^4=(x^2-y^2)(x^2+y^2)=(x-y)(x+y)(x^2+y^2)$;
- $x^4+y^4+x^2y^2=x^4+y^4+2x^2y^2-x^2y^2=(x^2+y^2)^2-x^2y^2=(x^2+y^2-xy)(x^2+y^2+xy)$;
- $x^4+y^4=x^4+y^4+2x^2y^2-2x^2y^2=(x^2+y^2)^2-2x^2y^2=(x^2+y^2-\sqrt{2}xy)(x^2+y^2+\sqrt{2}xy)$;
- $x^6-y^6=(x^3-y^3)(x^3+y^3)=(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$.
- $x^7+x^2+1=(x^7-x)+x+x^2+1=x(x^6-1)+x^2+x+1=x(x^3-1)(x^3+1)+x^2+x+1=x(x-1)(x^2+x+1)(x^3+1)+x^2+x+1=(x^2+x+1)[x(x-1)(x^3+1)+1]$

Other factorization formulae: $\boxed{x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)}$.

$\boxed{(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz.}$

$\boxed{(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_1x_2+y_1y_2)^2+(x_1y_2-y_1x_2)^2}$.

2. Polynomials in one variable x. Division with remainder.

Let $a(x)$ and $b(x)$ be two polynomials. Then there exists a unique pair of polynomials $q(x)$ (***the quotient***) and $r(x)$ (***the remainder***) for which

$$\boxed{a(x)=b(x)q(x)+r(x)} \text{ and } \text{ degree } r(x) < \text{ degree } q(x).$$

This is similar to the division algorithm for integers.

The ***greatest common divisor*** $d(x)$ of two polynomials $a(x)$ and $b(x)$ is a polynomial of largest degree which divides exactly into both $a(x)$ and $b(x)$. It can be found by repeated division (Euclid's algorithm).

Example: Find the greatest common divisor of x^3+x^2-3x-6 and x^3-3x-2.

By division we get

$$\begin{aligned}\boxed{x^3+x^2-3x-6} &= \boxed{(x^3-3x-2)}+\boxed{(x^2-4)}\\ \boxed{x^3-3x-2} &= \boxed{(x^2-4)}x+\boxed{(x-2)}\\ \boxed{x^2-4} &= \boxed{(x-2)}(x+2)+0\end{aligned}$$

so $x-2$ divides x^2-4 and hence also x^3-3x-2 and hence also x^3+x^2-3x-6. The greatest common divisor is $x-2$.

The Remainder Theorem: Let a be a constant. Dividing a polynomial $P(x)$ by $(x-a)$ yields the number $P(a)$ as remainder:

$$\boxed{P(x)=(x-a)Q(x)+P(a).}$$

Proof:

Indeed, since $(x-a)$ has degree 1, the remainder must have degree 0 and hence be a number, let's call it R.

$$\boxed{P(x)=(x-a)Q(x)+R.}$$

Set $x=a$ to get $p(a)=r$.

So $(x-a)$ is a factor of $P(x)$ if and only if $P(a)=0$.

Example: Factor $P(x)=x^3-5x^2+3x+1$.

Solution: We search for an integer number a such that $P(a)=0$. Then $a^3-5a^2+3a+1=0$ so $a^3-5a^2+3a=-1$. Since the left-hand-side is a multiple of a, then a must be a divisor of -1.

We try $a=1$. Then $P(1)=1-5+3+1=0$. The remainder theorem implies that $P(x)$ has $x-1$ as factor. We can divide $P(x)$ by $x-1$ as follows:

We note that for any number k we have $x^k(x-1)=x^{k+1}-x^k$ and $-x^k(x-1)=-x^{k+1}+x^k$. So we can try to write $P(x)$ as a sum of terms of these forms:

$P(x)=x^3-x^2-4x^2+4x-x+1=x^2(x-1)-4x(x-1)-(x-1)=(x-1)(x^2-4x-1)$. By completing the square:

$P(x)=(x-1)(x^2-4x+4-5)=(x-1)[(x-2)^2-5]=(x-1)[(x-2)^2-\sqrt{5}^2]=(x-1)(x-2-\sqrt{5})(x-2-\sqrt{5})$.