PIGEONHOLE PRINCIPLE

Socks

- You have a drawer full of black and white socks. Without looking in the drawer, how many socks must you pull out to be sure that you have a pair of the same colour?
- Trying to spice up your life, you go and buy a large number of red socks, green socks, and blue sock. Now how many must you pick out to be sure of having a pair of the same colour?
- Now, being of a mathematical persuasion, you have n different coloured socks in your drawer. Now how many must you pick to be sure of having a pair of the same colour?

Pigeonhole Principle

- If you place $n + 1$ objects in n holes, then at least one hole must contain more than one object.

9 holes, and 10 = 9 + 1 pigeons.

So at least 1 hole contains at least 2 pigeons.

More Socks

- A dog-owner has a drawer with her dog-socks in it. Each sock is either black or white. Presuming the dog has 4 legs, each of which needs a sock, how many must be taken from the drawer to ensure that the dog has 4 socks of the same colour?
- What if there are dog socks in all the colours of the rainbow, plus black and white – how many must be picked now to guarantee 4 socks of the same colour?
- Paul the Octopus has a drawer with socks, each of which is a colour featured on the German flag. How many socks must he pick in order to have 8 of the same colour?

Generalised PP

- If you have n boxes, and $n\,k + 1$ objects, then at least one box contains more than k objects.

Examples

1. How many people must be assembled to ensure that 10 of them were born on the same day of the week? (EGMO pre-selection test 2012)
2. Suppose 251 numbers are chosen from 1, 2, 3, … , 500. Show that, no matter how the numbers are chosen, you must have 2 that are consecutive. (EGMO selection test 2012)
3. Seven darts are thrown at a circular dartboard of radius 10cm. Show that there will always be two darts that are at most 10cm apart. (EGMO selection test 2012)
4. A doctor has a row of 16 chairs in her waiting room. What’s the least number of patients she must have to ensure that three consecutive chairs are filled?

General Comment

- The PP sounds pretty vague – it doesn't tell you exactly how many objects are in any given box, it just gives a minimum for the number of objects in the fullest box.
- For example in Q1, you can't say what day the 10 birthdays must be on, or whether there are more than ten birthdays on that day; you can only say that there must be some day with at least 10 birthdays on it.

General Comment

- The conclusion of the PP must be true no matter how the objects are arranged inside each box, as long as you have the right number of objects. It is often not enough to consider just 1 "worst case scenario" when just 1 object is missing, because there might be many different arrangements even among the "worst case scenarios".

- For example in Q4, all these 3 cases can be considered "worst case scenarios":

L L L L L L L L L L L L L L L L
L L L L L L L L L L L L L L L L
L L L L L L L L L L L L L L L L

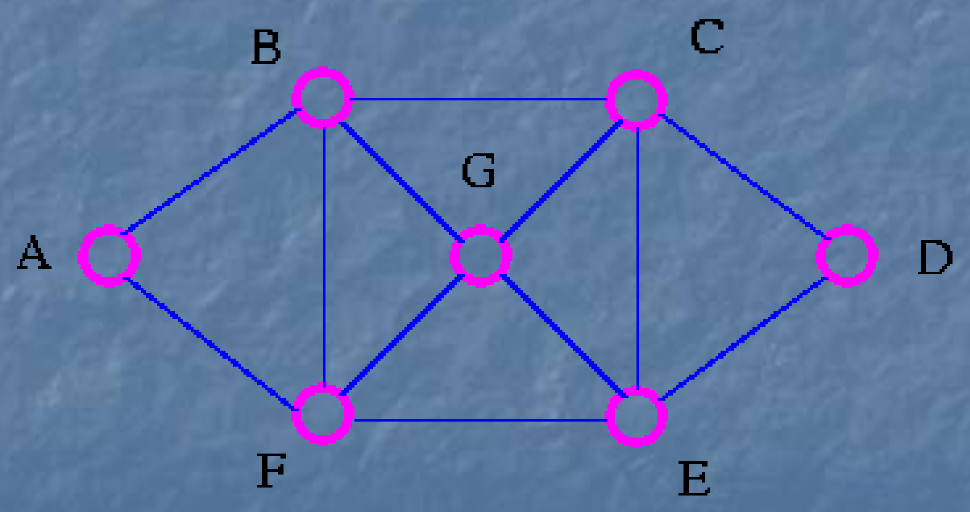
We don't know which of these the patients will choose.
Choosing suitable boxes helps you treat all cases at once.

More Examples

5. Show that at a party with 6 people, there are two people who know the same number of people at the party. Assume that "knowing someone" is a symmetric relation, i.e. if *x* knows *y* then *y* knows *x* (though may not be able to remember his name, which can be embarrassing).
6. Given any 13 integers, show that there is a pair whose difference is divisible by 12.
7. Given any 8 integers, show there is a pair whose sum, or difference, is divisible by 12.

Euler Circuits

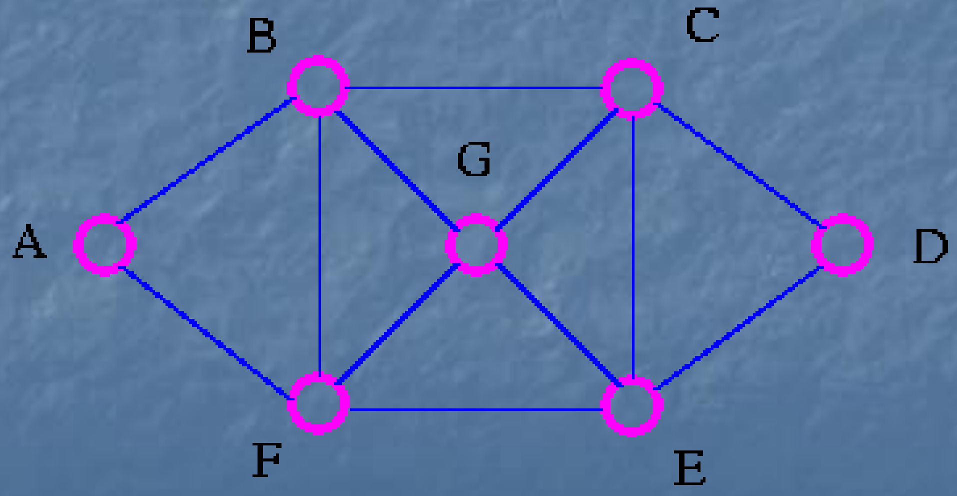
- Can be drawn without raising the pen off the paper, without tracing over the same edge twice, and finishing where we started.



Euler Circuits

- Can be drawn without raising the pen off the paper, without tracing over the same edge twice, and finishing where we started.
- Each vertex must have an even number of neighbouring edges, because whenever you enter a vertex by one edge, you must exit by another edge.

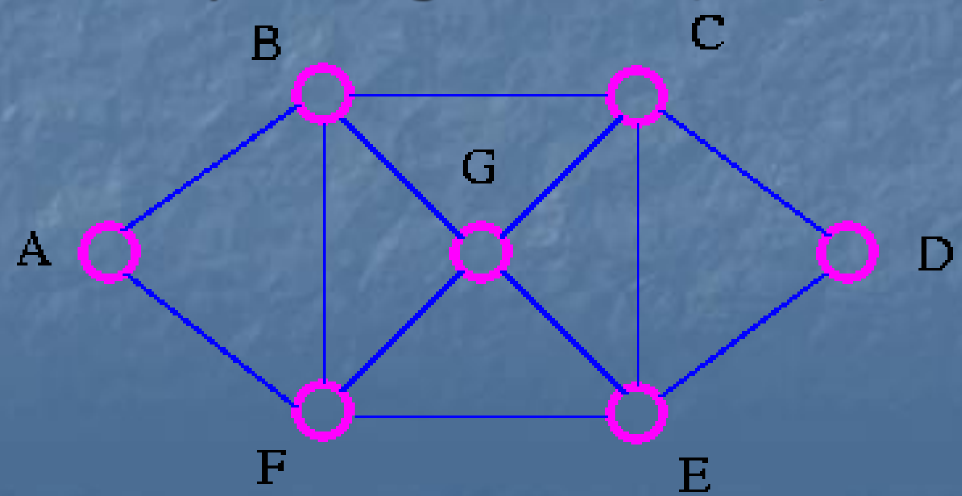
deg(A)=2
deg(B)=4
deg(C)=4
…………….



Application to Finances

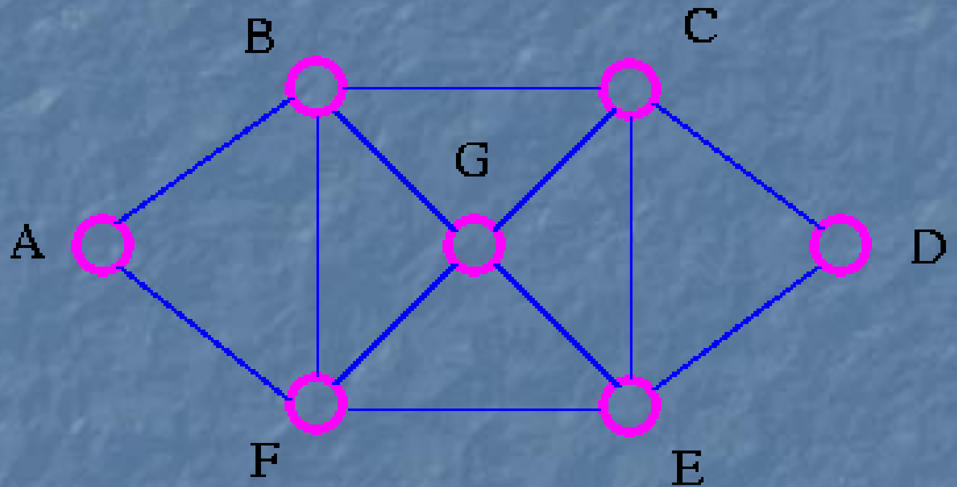
Each vertex represents a bank. Each edge AB represents a financial transaction: A client of bank A draws a mortgage of €200K and buys a house from a client of bank B, who deposits the €200K in his/her account in bank B. Because we're in an Euler circuit, at the end of the day, none of the banks has spent any money.

For each mortgage, the client owes the bank €200K + an interest of €100K payable over 20-30 years. In total, the 12 transactions will yield a gain of €1,200,000 in interest.



Application to Finances

When a small set of players in an economy have unrestricted ability to create free money, this will lead to inequality. When the same players make decisions as to where to invest their money, this will lead to economical imbalances. Unrestricted, such a system will lead to crises.



Banks and people are clearly on unequal terms. But how do the banks in an Euler circuit compare with each other?

Q: Prove that for every simple Euler circuit, there always exist at least three vertices with the same degree.

In our problem, at least three banks with gain the same amounts.