Modular Arithmetic

(1) Find the remainder of
   a) 2^{2013} when divided by 3.
   b) 3^{2013} when divided by 5.
   c) 3^{1945} when divided by 21.

(2) Find the remainder when dividing $a = 1^{2013} + 2^{2013} + 3^{2013} + ... + 2013^{2013}$ by 5.

(3) Square each of the first 15 natural numbers and find its last digit. Do you notice a pattern? Can you explain it?

(4) Consider the number $x = 50 \cdot 5^{2010} \cdot 2^{2011} - 2012$.
   a) Is x a perfect square?
   b) Find the sum of all the digits of the number x.

(5) Show that the sum of squares of 5 consecutive numbers cannot be a perfect square.

(6) Consider the number 11111...1 written with 2011 of 1-s. What is its remainder when divided by 3? by 9? by 7? by 11?

(7) If n is any integer number, prove that:
   a) $n(n+1)(n+2)$ as well as $n(n+1)(2n+1)$ are divisible by 6.
   b) $17n^2 + 1$ is divisible by neither 4 nor 5.
   c) $n^5 + 29n$ is divisible by 30.
   d) $n^2 - 1$ is always divisible by 24, provided that neither 2 nor 3 divides n.

(8) a) If $3|(a^2 + b^2)$, prove that $3|a$ and $3|b$.
   b) If $5|(a^2 + b^2 + c^2)$, prove that $5|a$ or $5|b$ or $5|c$.

(9) Let a, b and c be integer numbers such that 6 divides $a + b + c$. Show that 6 also divides $a^5 + b^3 + c$.

(10) Let a, b, c be natural numbers. If $2a^3 - 3a^2b + 2b^3$ is divisible by 5, show that a and b are divisible by 5.

(11) If p is prime, prove that $p^2 + 2$ is composite.

(12) Find all prime numbers p such that $p^2 - 2$, $2p^2 - 1$ and $3p^2 + 4$ are also primes. (Hint: think of the possible remainders of p when divided by 7).

(13) a) Show that there are infinitely many prime numbers of the form $3k + 2$, where k is integer.
   b) Same for $4k + 3$, where k is integer.
   c) Same for $6k + 5$, where k is integer.

(14) Show that among 12 consecutive natural numbers there exists one smaller than the sum of its divisors (not including 1 and itself).

(15) Find the smallest positive integer congruent to 2 mod 3, to 4 mod 5, and to 6 mod 7.
   a) Find all solutions of the equation $x^6 + x + 3 \equiv 0 \bmod 5$.
   b) Find all solutions of the equation $x^6 + x + 3 \equiv 0 \bmod 25$.
   c) Find all solutions of the equation $x^6 + x + 3 \equiv 0 \bmod 75$.

(16) The first 8 terms of a certain sequence are given below.

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad ...$$

The sequence follows a certain mathematical rule. In your opinion,
   a) What should the next 8 terms in the sequence be?
   b) The 2nd, the 5th, the 8th numbers above are even. If the sequence goes on forever, can you describe all numbers N such that the N-th number in the sequence is even?

c) What is the remainder when the 2012-th term of the sequence is divided by 4?

d) The 4th number in the sequence is a multiple of 5. Prove that the multiples of 5 in the sequence are exactly the 4-th, 9-th, 14-th,...,$(5k+4)$-th numbers for all natural numbers k.

e) Based on the sequence above, we construct the following:

$$1, \quad 11, \quad 111, \quad \underbrace{11111}_{5 \text{ ones}}, \quad \underbrace{11111111}_{8 \text{ ones}}, \quad \underbrace{1111111111111}_{13 \text{ ones}}, \quad \underbrace{111111111111111111111}_{21 \text{ ones}}, \quad ...$$

Prove that any two numbers in the sequence which are either neighbors or separated by just one other member of the sequence are coprime, meaning that their largest common divisor is 1.