Mathematical Games

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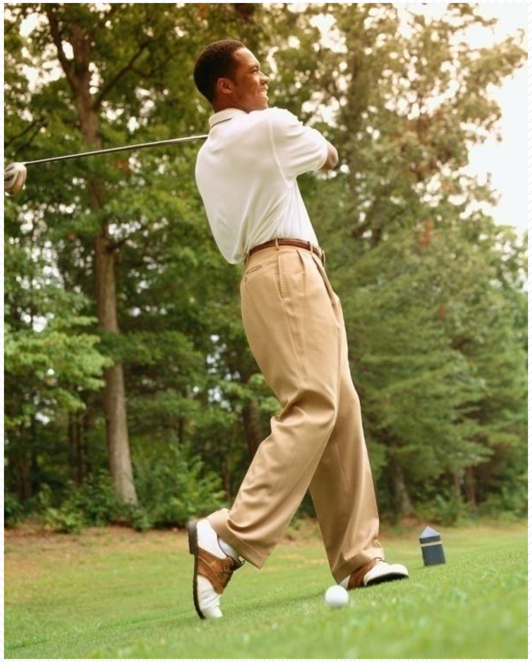
One Player Games



Bored MUMSians

- Han writes the numbers $1, 2, \ldots, 100$ on the whiteboard. He picks two of these numbers, erases them, and replaces them with their difference (so at some point in time we may have replications of a number). He repeats the process until he is left with just one number.

- Is it possible that he is left with the number 1?

- The sum of all of the numbers on the whiteboard is initially 5050 (exercise!), which is even.
- With each ‘move’, the sum decreases by: a + b - |a - b|.
- This is always even (exercise!).
- Therefore the sum of all numbers on the whiteboard is at any stage even (it begins even, and then it decreases by an even amount with each move).
- So Han cannot be left with the number 1 (because then the sum would be odd).

Invariants

- An invariant is something which does not change no matter what happens.
- In the previous example, the parity of the sum of the numbers on the whiteboard was invariant whenever Han made a ‘move’; it remained even.

The Classic Example

- Take a chessboard (8x8), and remove two opposite corners.
- Can you tile what's left with dominoes (1x2 blocks)?

A Simple Colouring

- Colour the chessboard in the usual manner.
- Now we have 32 black squares, but only 30 white squares.
- However each domino covers one white square and one black square.
- So we cannot cover 32 black squares, since we only have 30 white squares.

An Extension

- You place 21 trominoes (1x3 blocks) on a chessboard (8x8), so that there is one square not covered.
- What are the possible positions for this square?

Two Colourings!

R	B	G	R	B	G	R	B
G	R	B	G	R	B	G	R
B	G	R	B	G	R	B	G
R	B	G	R	B	G	R	B
G	R	B	G	R	B	G	R
B	G	R	B	G	R	B	G
R	B	G	R	B	G	R	B
G	R	B	G	R	B	G	R

B	R	G	B	R	G	B	R
R	G	B	R	G	B	R	G
G	B	R	G	B	R	G	B
B	R	G	B	R	G	B	R
R	G	B	R	G	B	R	G
G	B	R	G	B	R	G	B
B	R	G	B	R	G	B	R
R	G	B	R	G	B	R	G

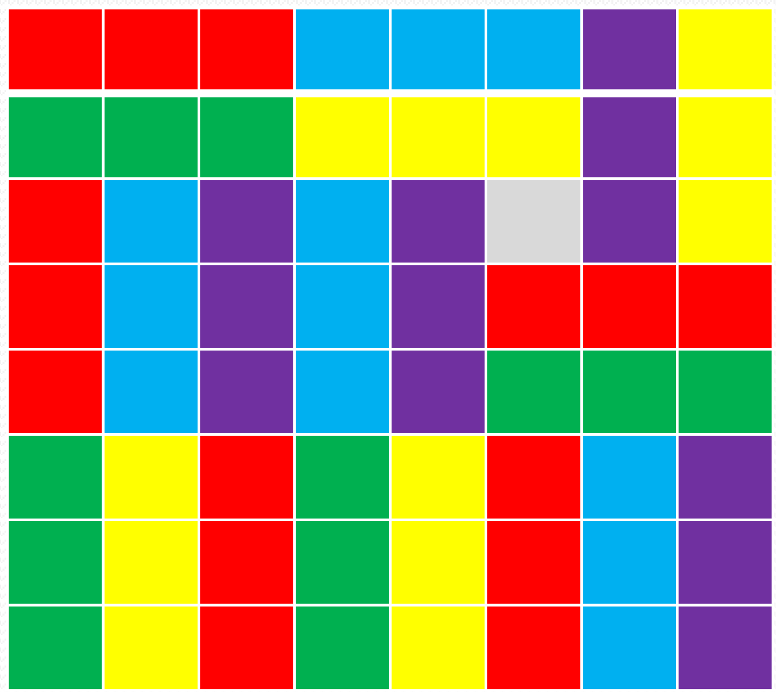
- In both diagrams, each tromino must cover one red square, one blue square, and one green square.
- In both diagrams, there are 22 red squares, but only 21 blue and green ones.
- So the remaining square must be red in both the first and second diagrams, giving 4 possibilities:

Four Possibilities

		R			R		
		R			R		

Final Step

- We still have to demonstrate that these are possible.
- By symmetry, we only have to show that one of them is possible:

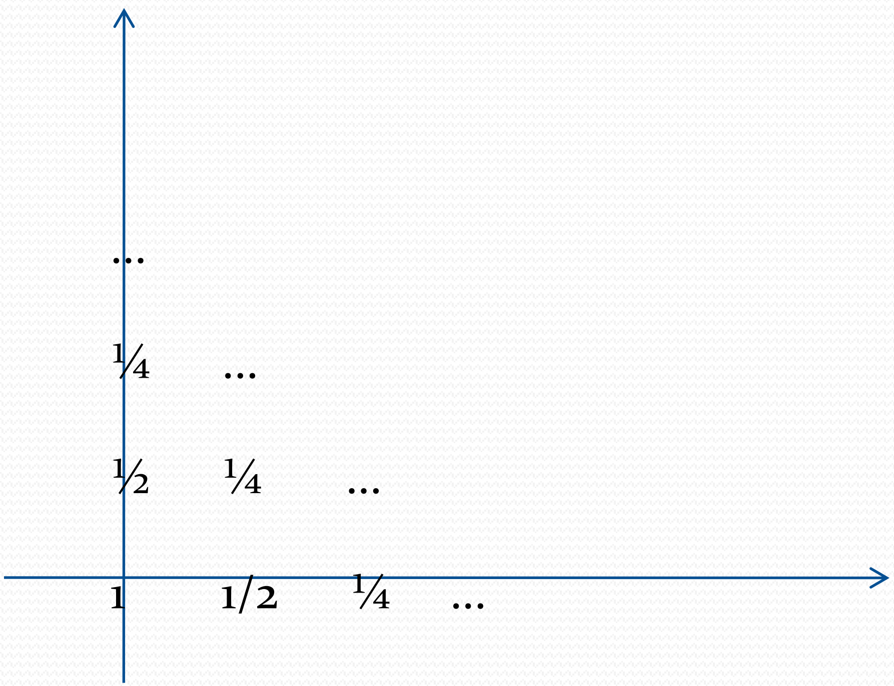


Amoebae

- Three amoebae are situated at the points (0, 0), (0, 1) and (1, 0) of the Cartesian plane. Each second, one of the amoebae reproduces by **splitting** into two new amoebae. If the parent amoeba was at (x, y), then its offspring will begin at (x, y+1) and (x+1, y). No two amoebae may occupy the same position. Is it possible to vacate (in a finite amount of time) the original three points in a finite amount of time?

Valuing the Points

- We assign the points values as follows:



The Invariant

- Let X be the sum of the values of the points on which the amoeba lie.
- Initially X = 2.
- X is invariant; it does not change when an amoeba splits.
- The total value of all points is 4 (see board).
- The total value of all points apart from the initial 3 points is 2.
- If we were to vacate the initial 3 points, then – in order for X to equal 2, every other point would have to be occupied, which is **impossible**.

Atypical Colourings

- In the table below, you may switch the signs of all numbers of a row, column or a parallel to one of the diagonals. In particular, you may switch the sign of each corner square. Prove that at least one -1 with always remain.

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

- Every such 'line' goes through either 0 or 2 (an even number of) red squares, so the parity of **the number of '-1's in red squares** will not change!
- Initially it is 1, so it must stay odd. In particular it can never become 0.

My Question

- Asian Pacific Mathematical Olympiad 2007, Q5.

- A regular 5x5 array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on.
- Find all the possible positions of this light.

First Valuation System

- We assign the following **values** to the lights.
- Let x be the sum of the values of the lights that are switched on at a particular point in time.
- By inspection, toggling any switch does not change the parity of x.

1	1	0	1	1
0	0	0	0	0
1	1	0	1	1
0	0	0	0	0
1	1	0	1	1

Second Valuation System

- Now assign new values to the lights (actually we've just rotated the previous diagram!)
- Let y be the new sum.
- Toggling any switch does not change the parity of y.

1	0	1	0	1
1	0	1	0	1
0	0	0	0	0
1	0	1	0	1
1	0	1	0	1

5 Possibilities

- So the parities of x and y are invariant.
- Initially x = y = 0, so x and y must always be even.
- The light that's switched on at the end must have value 0 under both valuation systems (it cannot be 1, since x and y must both be even).
- 5 possibilities:

	x		x	
		x		
	x		x	

Constructions

- By rotating the second sequence of toggles, we can attain the other three positions.
- So all 5 possibilities were indeed possible.

			T	T
		T		
	T	T		T
T				T
T		T	T	

T		T	T	T
T			T	
	T			
	T	T	T	
T	T	T	T	T

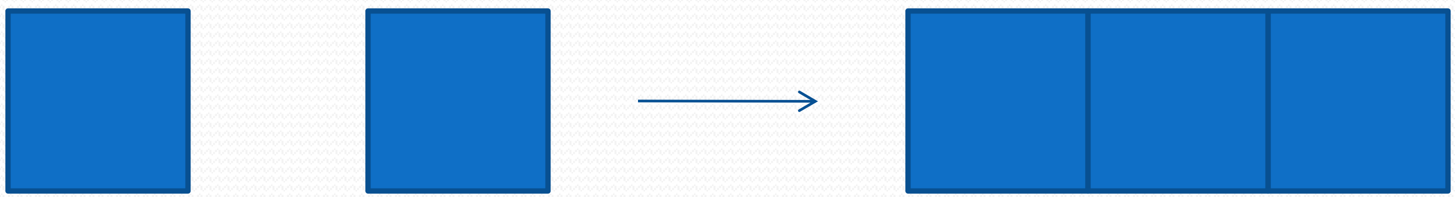
Something Different

- Nine 1x1 cells of a 10x10 square are infected. At the end of each second, any uninfected cell with at least two infected neighbours (having a common side) becomes infected. Once infected, a cell remains infected. Can the infection spread to the whole square?

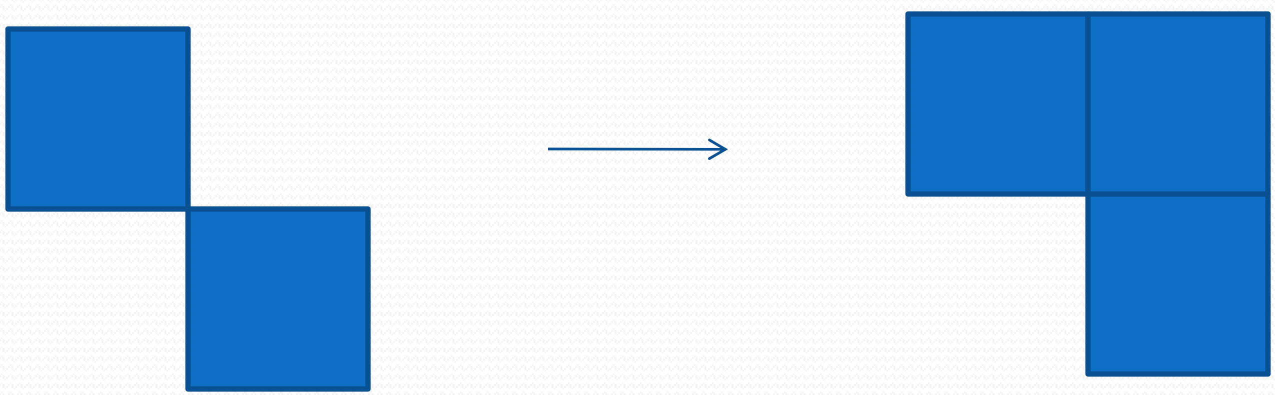
A Monovariant

The perimeter of the infected squares cannot increase!

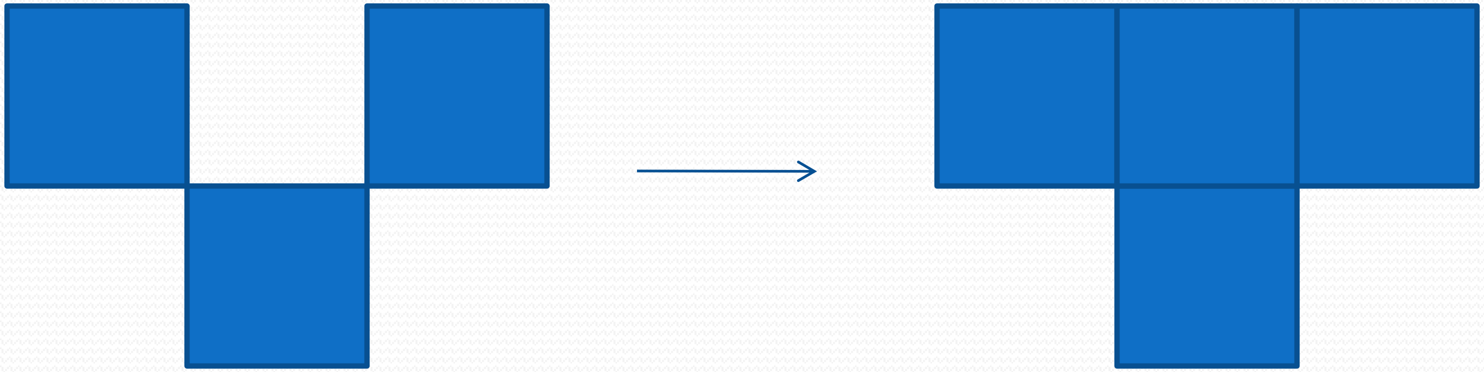
E.g. 1



E.g.2



- ...but it may decrease



- Initially the perimeter is at most 36 (= 9x4), and it cannot increase.
- So in particular, it cannot eventually become 40.
- So the infection cannot spread to the whole square.

Two Player Games

Definition

A *combinatorial game* is one in which:

- there are two players
- there is no luck involved
- both players know the position of the game at all times
- there is a result (win, loss, or draw)
- the game does not go on forever

Crucial Fact

In a combinatorial game, one of the following **must** be true:

- The first player has a winning strategy.
- The second player has a winning strategy.
- Either player can force a draw.

Double Chess

- The rules of chess are changed as follows: each player gets two moves instead of one. Show that there exists a strategy for white which guarantees him at least a draw (remember that white starts first).

Proof by Contradiction

- **Suppose** white has no strategy which guarantees at least a draw.
- Then a winning strategy **must** exist for black.
- White can move his knight out and then back, and black must still have a winning strategy from here.
- However, black is in the same position white was in to begin with!
- The implication is that white had a winning strategy to begin with. Contradiction.
- Hence our initial assumption is false. Instead, white must have a strategy which guarantees at least a draw.

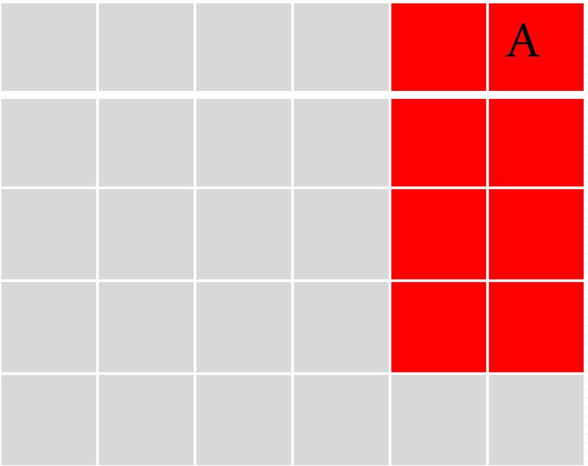
Chomp

- We have an m x n block of chocolate. A move consists of eating a piece of chocolate, along with every piece of chocolate above and to the right of it. Players move alternately, and the player who eats the bottom-left piece loses (it's poisoned).
- Play with the person next to you.

Existence

- We may not know what it is (I actually do not), but in fact we can prove that the first player has a winning strategy!
- Again we will use proof by contradiction.
- **Suppose** the second player has a winning strategy.
- **If** the first player bites off the top right corner (call this move A), the second player must have **some** winning move: call it B.

Strategy Stealing



- However, no matter what B actually is, player 1 could have played B to begin with!
- Now player 1 has a winning strategy, **contradicting** our assumption that player 2 has one.
- So our initial assumption is false: player 2 cannot have a winning strategy, so player 1 must have a winning strategy (since there are no draws).

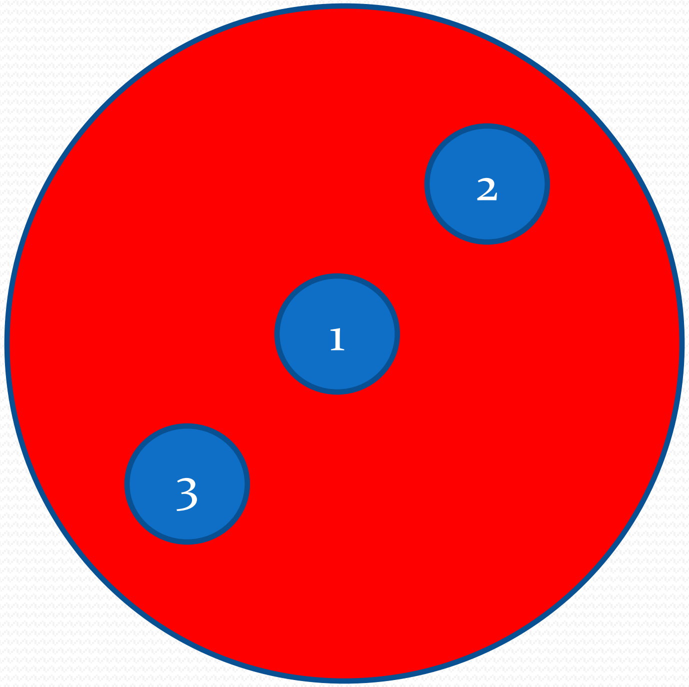
Yum Cha

- In the World Yum Cha Championships the final two waitresses ‘face-off’ in a match for eternal glory. A perfectly circular ‘lazy Susan’ stands in the middle of a table, and the two waitresses take turns placing a plate onto the lazy Susan (the plates are small, circular, and identical). Plates may not overlap (you can’t just stack them). Whoever is unable to legally place a plate loses. One year, however, the two final waitresses are both infinitely skilful! Who wins?

Copycat

- The first waitress confidently places a plate right in the middle of the lazy Susan.
- Wherever the second waitress puts a plate, the first waitress places one diametrically opposite.
- Whenever the second waitress has a legal placement, so does the first waitress!
- So eventually it will be the second waitress who runs out of legal 'moves'.

- The first waitress plays at 1, the second plays at 2, and the first plays at 3. The first waitress continues to ‘copy’ the second.



Infinite Naughts and Crosses

- There is an infinitely long line of squares. Two players take turns placing their symbol in one of the squares. The first player to make three in a row wins. Show that neither player can force a win.
- This is the same as showing that either player can force a draw.

Grouping

- We make pairs of adjacent squares:

- If you're player 2, then you just always play in the same pair as player 1.
- If you're player 1, then you play somewhere random. If player 2 plays in a 'new' pair, then you 'finish off' that pair.
- If he finishes a pair you started, then you play somewhere random again.
- So both players can force a draw.

Chip

- Initially, there is a chip at the corner of an n x n chessboard. Two players, A and B, alternately move the chip one step in any direction (up, down, left, right). When the chip leaves a square, it disappears. The loser is the one who cannot move.

- For which values of n can player A win the game?

Case 1: n is odd.

6	7	7	8	X
6	11	12	8	1
5	11	12	9	1
5	10	10	9	2
4	4	3	3	2

- We pair up the remaining squares.
- Wherever A plays, B plays in the other square of the pair.
- If A has a legal move, then so will B.
- Eventually A will run out of moves, and B will win.

Case 2: n is even.

15	14	9	6	5	X
15	14	9	6	5	0
16	13	10	7	4	1
16	13	10	7	4	1
17	12	11	8	3	2
17	12	11	8	3	2

- A plays at 0, and we pair up the remaining squares (there are many ways to do this).
- A follows the same strategy as B followed in the odd case, so that this time A wins.

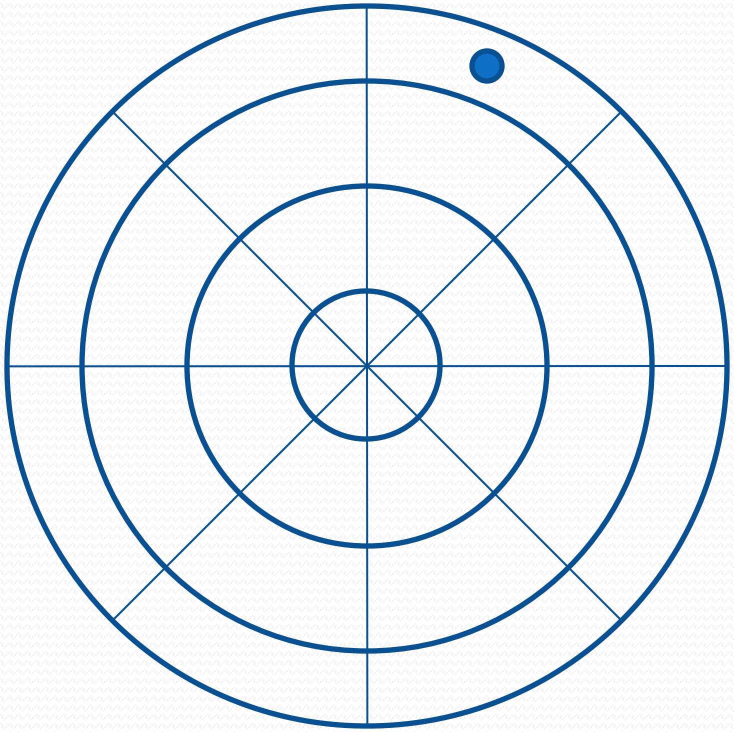
The 31 Game

- The total starts at 0.
- Two players take turns to add a number (1, 2, 3, 4 or 5) to the total.
- The player who reaches 31 wins.
- Who wins, A or B?

- A adds 1, bringing the total to 1 (to see that this is the correct first move, notice that 31 is congruent to 1 modulo 6).
- If B adds x, A will add 6-x, bringing the total to 7.
- The process repeats, bringing the total (after A moves) to 13, 19, 25, and finally 31.
- A wins.

Token

- University of Melbourne (BHP Billiton) Mathematics Competition 2008 (Junior), Q8?



Interesting Choices

- Two players take turns to move the token in one of three ways:
  - Inwards one
  - Clockwise 1
  - Clockwise 1 twice (as distinct from clockwise 2 – see next point)
- You cannot move the token where it has been before.
- You lose if you cannot move.
- Prove that the first player can force a win.

Sketch of Proof

- A new ring has been entered (by B, or initially).
- A plays clockwise once, leaving 6 spots in that ring.
- B can enter a new ring (in which case we move onto the next ring and do the same thing), or he can go clockwise 1 or 2.
- A responds to the latter 2 by moving clockwise 2 or 1 (respectively), leaving 3 spots in the ring.
- If B doesn't enter a new ring, A will use the same trick so that there are no spots in the ring.

Ring Out

- If we aren't in the inner ring, B will have to be the one who moves inwards.
- If we are in the inner ring, B will run out of moves and lose.

- Again, we motivate the first move by observing that 8 is congruent to 2 modulo 3.

Number Game

- Two players start with N = 1.
- They alternatively multiply N by one of the numbers 2 to 9.
- The winner is the first to reach 1000 or more.
- Who wins?

Positions of Win (or Loss)

- Anything from 112 to 999 (inclusive) is a winning position (W), since the player facing that position **can** win by multiplying by 9.
- Anything from 56 to 111 is L, since you **must** put your opponent in a W position.
- Anything from 7 to 55 is now W, since you **can** move to an L position.
- 4, 5, 6 are L.
- 1 is W.

Summary of A's Strategy

- A multiplies N by 4 to give N=4.
- B moves N somewhere between 7 and 55.
- A moves somewhere between 56 and 111.
- B moves somewhere between 112 and 999.
- A moves somewhere over (equal to suffices) 1000 and wins.

Nim

- There are a number of piles of sticks.
- Two players take turns to remove any number (at least one) from a pile.
- In ‘normal play’, the winner is the player who takes the last stick.
- In misère Nim, the player who takes the last stick loses.
- Nim has been solved!

Xor

- The crucial value is the **binary digital sum** of the pile sizes, which is known in combinatorial game theory as the **Nim-sum.**
- This operation is perhaps better known as ‘exclusive or’ (xor).
- Write the numbers out in binary, and add them (in binary) ‘without carrying forward’ from one digit to another.
- See board for example.

Lemmas of Win

- In **normal play**:
  - Positions of non-zero Nim-sum are W.
  - Positions of zero Nim-sum are L.
- Need the following lemmas (C. Bouton):
  - If the Nim-sum is zero, any move will make it nonzero.
  - If the Nim-sum is nonzero, there is a move that will make it zero.
- So the winning strategy is to make the Nim-sum zero with every move.