Inequalities and Applications

1. The AM-GM-HM Inequality

Given n **positive** numbers $x_1 > 0, x_2 > 0, ..., x_n > 0$, where $n \geq 2$, define:

The Arithmetic Mean: $\boxed{AM = \dfrac{x_1 + x_2 + ... + x_n}{n}}$.

The Geometric Mean: $\boxed{GM = \sqrt[n]{x_1 x_2 \cdot ... \cdot x_n}}$.

The Harmonic Mean: $\boxed{HM = \dfrac{n}{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}}}$.

The AM-GM-HM Inequality:

$$\boxed{AM \geq GM \geq HM},$$

with equality reached **when all numbers are equal**.

For example, when $n = 2$ and the two numbers are x and y we have

$$AM = \frac{x+y}{2},\ GM = \sqrt{xy},\ HM = \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

The reason why $AM \geq GM$ in this case is:

$$\boxed{(\sqrt{x} - \sqrt{y})^2 \geq 0} \iff x + y - 2\sqrt{xy} \geq 0 \iff x + y \geq 2\sqrt{xy}.$$

1. Prove the AM-GM-HM Inequality in general following these steps:
a) Use *Induction* on k to prove the AM-GM Inequality in the case $n = 2^k$.

b) If $n < 2^k$, apply the AM-GM Inequality to the sequence of numbers

$$x_1, x_2, ..., x_n, \underbrace{AM, AM, ..., AM}_{2^k - n \text{ times}}.$$

Use this to prove the AM-GM Inequality for $x_1, x_2, ..., x_n$.

c) Prove that the GM-HM Inequality for $x_1, x_2, ..., x_n$ is equivalent to the AM-GM Inequality for $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$.

2. Each of the 60 TY students in a school has read 1, 2 or 3 of the Hunger Game books. If a team of 3 students is to be chosen for a quiz, such that one member has read 1 book, the 2nd member has read 2 and the third has read 3 books, what is the largest possible number of choices for such a team?

2. The Cauchy-Schwartz Inequality

Given $2n$ numbers $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$, then

$$\boxed{(x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \geq (x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2}$$

3. Let $S = (x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) - (x_1y_1 + x_2y_2 + ... + x_ny_n)^2$

a) In the case $n = 2$, check that $S = (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1y_1 + x_2y_2)^2 = (x_1y_2 - x_2y_1)^2 \geq 0$.

b) In general, show that that $S = \sum_{i \neq j} (x_iy_j - x_jy_i)^2 \geq 0$. Here "$\sum_{i \neq j}$" stands for the sum of all terms when $i \neq j \in \{1, 2, ..., n\}$.

4. If x_1, x_2, x_3 are three positive numbers such that $x_1 + 2x_2 + 3x_3 = 60$, what is the smallest possible value of the sum $x_1^2 + x_2^2 + x_3^2$?

3. Quadratic Inequalities

The source of all quadratic inequalities is:

$$\boxed{(x-y)^2 \geq 0} \Longleftrightarrow x^2 + y^2 - 2xy \geq 0 \Longleftrightarrow \boxed{x^2 + y^2 \geq 2xy}, \tag{3.1}$$

We will also use the related formula:

$$\boxed{(x+y)^2 = x^2 + y^2 + 2xy}.$$

5. Knowing that two numbers x and y satisfy $x + y = N$, where N is a given number which we will assume known.

a) What is the largest possible value of xy? Answer: $\boxed{xy \leq \frac{(x+y)^2}{4}}$.

Idea of proof: You start from the inequality

$$x^2 + y^2 \geq 2xy.$$

You're interested in how big the RHS (right hand side) can get. So you like the RHS, but not the LHS – you want to change it to some expression whose value you know. For this, add $2xy$ to both sides:

$$\begin{aligned} x^2 + y^2 + 2xy &\geq 4xy \\ (x+y)^2 &\geq 4xy. \\ \frac{N^2}{4} = \frac{(x+y)^2}{4} &\geq xy. \end{aligned}$$

b) What is the smallest possible value of x^2+y^2? Answer: $\boxed{x^2 + y^2 \geq \frac{(x+y)^2}{2}}$.

c) What is the smallest possible value of $\frac{1}{x^2} + \frac{1}{y^2}$?

d) If both x and y are positive, what is the largest possible value of $\sqrt{x} + \sqrt{y}$?

We will generalize this to n numbers $x_1, x_2, ..., x_n$. For each pair of two such numbers, we write the basic formula:

$$\begin{array}{rcl} x_1^2 + x_2^2 & \geq & 2x_1x_2; \\ x_1^2 + x_3^2 & \geq & 2x_1x_3; \\ \dots\dots\dots & \dots & \dots\dots\dots \\ x_1^2 + x_n^2 & \geq & 2x_1x_n; \\ x_2^2 + x_3^2 & \geq & 2x_2x_3; \\ \dots\dots\dots & \dots & \dots\dots\dots \\ x_{n-1}^2 + x_n^2 & \geq & 2x_{n-1}x_n; \end{array}$$

Add up all terms: $\boxed{(n-1)(x_1^2 + x_2^2 + ... + x_n^2) \geq 2(x_1x_2 + x_1x_3 + ... + x_{n-1}x_n)}$.

The other useful formula:

$$\boxed{(x_1 + x_2 + ... + x_n)^2 = x_1^2 + x_2^2 + ... + x_n^2 + 2x_1x_2 + 2x_1x_3 + ... + 2x_{n-1}x_n.}$$

6. Knowing that $x_1, x_2, ..., x_n$ satisfy $x_1 + x_2 + ... + x_n = N$, where N is a given number which we will assume known.

a) What is the largest possible value of $x_1x_2 + x_1x_3 + ... + x_{n-1}x_n$?

Answer: $\boxed{x_1x_2 + x_1x_3 + ... + x_{n-1}x_n \leq \frac{n-1}{2n}(x_1 + x_2 + ... + x_n)^2}$.

b) What is the smallest possible value of $x_1^2 + x_2^2 + ... + x_n^2$?

Answer: $\boxed{x_1^2 + x_2^2 + ... + x_n^2 \geq \frac{1}{n}(x_1 + x_2 + ... + x_n)^2}$.

d) If $x_1, x_2, ..., x_n$ are positive, what is the largest possible value of

$$\sqrt{x_1} + ... + \sqrt{x_n}?$$

4. Applications to Graph Problems

7. There are N points on the plane. If you connect each pair of points by a segment, how many segments do you get?

8. In a chess competition, a group of $N = 100$ competitors are to form $n = 10$ teams, and each member of a team will only play against the members of the other teams, once with each. What is the largest possible number of games in such a competition, if teams are allowed to have any numbers of members?

9. In the 1st Round of a chess competition, a group of $N = 100$ players are split into $n = 10$ divisions, such that all people from the same division play together exactly once, while people from different divisions don't play together. What is the smallest number of games that can be played, and how should the players be split into divisions to insure that number?

10. There are $n = 10$ people in a room, of which $e = 30$ pairs of people who know each other. Two people are called indirect acquaintances if they both know a third person in the room.

a) For each person i in the room, let d_i be the number of people known by i. What is the sum of all d_i-s?

b) For each person i in the room, what's the maximum possible number of indirectly acquainted pairs who both know i? Write your answer as a function of d_i.

c) What is the smallest possible number of indirect acquaintances in the room?

11. In a chess tournament with n players, a total of m games were played. No pair of competitors played together more than once. A group of three players in which each pair played together during the tournament will be called a triangles.

a) For each player i, let A_i denote the set of all the competitors who played with i, and $|A_i| = d_i$. Prove that for two competitors i and j who have played together, the number of triangles containing the pair $\{i, j\}$ is

$$|A_i \bigcap A_j| \geq d_i + d_j - n.$$

b) Show that there are at least

$$\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$$

triangles in the competition.

12. Let $S = \{x_1, ..., x_n\}$ be a set of points in the plane such that the distance between any two points is at least 1. Show that there are at most $3n$ pairs of points at distance exactly 1.

[Hint: define a graph where the points at distance exactly 1 are joined by an edge. Prove that a point can be joined by at most 6 others.]

13. There are n blue points on the plane. Connect each pair of points found at a distance of exactly 1 cm from each others. For each point i, draw a circle C_i of radius 1 centered at the point.

a) What is the largest possible number of intersection points for all pairs of the circles drawn?

b) Let d_i be the number of points found at a distance of exactly 1 cm from i. How many pairs of circles intersect at i? Write your answer as a function of d_i. Add up to find the actual number of intersection points of all pairs of circles. Write your answer as a function of all d_i-s.

c) Prove that the number of pairs of points found at distance exactly 1 cm from each other is no larger than

$$\frac{n}{4}(1 + \sqrt{8n - 7}).$$