GETTING STARTED ON INEQUALITIES

(1) If x and y are natural numbers, find the smallest possible values of the following expressions:
 (i) $x^2 - 6x$.
 (ii) $3x^2 + y^2 + x - y$.
 (iii) $2x^2 + 5y^2 + 4xy - 8x - 14y$.
 Hint: (i) Write this quadratic in vertex form $(x-h)^2 + k$ by adding a suitable constant. Use the fact that squares are always non-negative. For which value of x is equality reached?
 (ii) By adding a suitable constant number, complete this to a sum $3(x+A)^2 + (y+B)^2 + C$ for suitable real numbers A, B and C.
 (iii) After adding a suitable constant, this can be written as $2(x+Ay+B)^2 + c(y+D)^2 + E$.

(2) Prove the following inequality for two positive numbers:
$$\frac{a+b}{2} \geq \sqrt{ab}.$$
This is called the $AM-GM$ inequality for two numbers. When are the two sides equal? Hint: After some algebra reduce this to the fact that the square of a difference of two numbers is non-negative.

(3) A piece of wire 40 cm long is bent to form the contour of a rectangle.
 a) Find (with proof!) the largest possible area of a rectangle thus formed.
 b) The rectangle thus formed is inscribed in a circle. Find (with proof!) the smallest possible radius of such a circle.

(4) A piece of wire 30 cm long is bent so as to form the perimeter of a right angled triangle. Find the largest possible area of such a triangle. When is it attained? Hint: Break the perimeter into a sum of two terms, and apply the AM-GM for each of these.

(5) Prove the $AM-GM$ inequality for four positive numbers:
$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$
When is equality attained? Hint: Break this into a sum of two sums, and apply the AM-GM (Q2) twice over.

(6) Prove the $AM-GM$ inequality for 2^n positive numbers. Hint: Induction using Q2.

(7) a) Prove the $AM-GM$ inequality for 3 positive numbers a, b, c by completing the sequence with a fourth number x. (Note: x should depend on a, b, c and should be equal to all when $a = b = c$).
 b) Prove the $AM-GM$ inequality in general:
$$\frac{x_1 + x_2 + \cdots x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$
 Hint: a) You can take $x = AM$ or $x = GM$. b) You can generalize the method used in a) and also use the result in the previous exercise.

(8) A cuboid of volume 8 cm ^{3} is to be gilded. It costs about 1 EU to cover 1 cm^2 in gold. What is the minimum possible cost for gilding such a cuboid?

(9) Use a change of variables $y_i = \frac{1}{x_i}$ and $AM-GM$ to prove $GM-HM$:
$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$
for positive numbers x_i.

(10) If $a, b, c \geq 0$, prove the following:
  (i) $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9$.
  (ii) $8abc \leq (a+b)(a+c)(b+c) \leq \frac{8}{27}(a+b+c)^3$.
  (iii) $abc(a+b+c) \leq a^4+b^4+c^4$.

(11) Prove the Cauchy-Schwartz Inequality

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

first for $n = 2$ and then in general. When is equality attained?
  Hint: Prove that $(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) - (\sum_{i=1}^n a_i b_i)^2 = \sum_{i<j}(a_i b_j - a_j b_i)^2$

(12) If $2x + 3y + 4z = L$ for some positive numbers x, y, z
  a) What's the largest possible volume of a cuboid with sides x, y and z?
  b) What's the shortest possible diagonal of a cuboid with sides x, y and z?

(13) If $x + y = L$ for some positive numbers x, y, what's the smallest possible value of $x^3 + y^3$?

(14) If $x+y+z = L$ for some positive numbers x, y, z, what's the smallest possible value of $x^2+y^2+z^2$?

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