CIRCLE GEOMETRY

Arcs and angles

Definition 1. An arc $\overset{\frown}{\mathrm{AB}}$ is a portion of a circle bounded by two points A and B on the circle.

Notation . For simplicity, we will identify the measure of an arc $\overset{\frown}{\mathrm{AB}}$ with that of the angle $\widehat{AOB}$, and write simply $\overset{\frown}{\mathrm{AB}} = \widehat{AOB}$. To justify this, we note that measuring the length of $\overset{\frown}{\mathrm{AB}}$ as a multiple of the radius will yield the same number as measuring $\widehat{AOB}$ in radians.

Lemma 2. ***(Angle on a circle.)*** *Let A, B, C be three points on a circle. Let $\overset{\frown}{BC}$ denote the arc which does not contain A. Then*

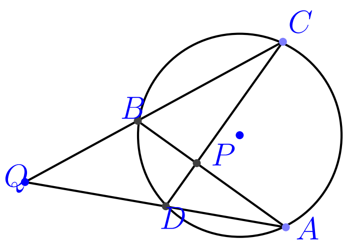
$$\widehat{BAC} = \frac{\widehat{BOC}}{2} = \frac{\overset{\frown}{BC}}{2}.$$

Proof. Let D be the point on the circle such that A, O and D are collinear. Then

- $\widehat{BOD} = 2\widehat{OAB}$, as exterior angle of the isosceles triangle OAB, and
- $\widehat{COD} = 2\widehat{OAC}$, as exterior angle of the isosceles triangle OAC.

If O is in the interior of the angle $\widehat{BAC}$, we add the equations above term by term, and O is outside of the angle $\widehat{BAC}$, we subtract the equations above term by term. In both cases, we obtain $\widehat{BOB} = 2\widehat{BAC}$. □

Corollary 3. ***(Internal and external angles)*** *Let $\mathcal{C}$ be a circle of center O and P a point in the interior of $\mathcal{C}$. Let A, B, C, D be points on $\mathcal{C}$ like in the figure here. Let Q be the intersection point of the lines AD and BC.*



Prove that

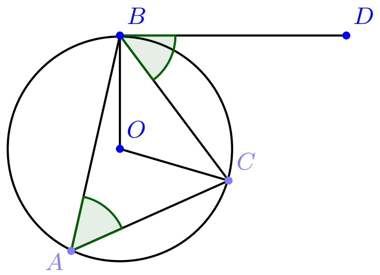
$$\widehat{APC} = \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD}) \text{ and } \widehat{AQC} = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{BD}).$$

Proof. $\widehat{APC}$ is exterior angle for $\triangle PBC \Longrightarrow \widehat{APC} = \widehat{PBC} + \widehat{PCB} = \frac{1}{2}(\overset{\frown}{\mathrm{AC}} + \overset{\frown}{\mathrm{BD}})$ (angles on the circle).

$\widehat{ABC}$ is exterior angle for $\triangle QBA \Longrightarrow \widehat{AQB} = \widehat{ABC} - \widehat{BAD} = \frac{1}{2}(\overset{\frown}{\mathrm{AC}} - \overset{\frown}{\mathrm{BD}})$. □

Lemma 4. *Let A, B, C be three points on a circle. Also, let BD be tangent to the circle, with D on the same side of AB as C. Then*

$$\widehat{BAC} = \widehat{DBC}.$$



The proof is left as an exercise.

Definition 5. A quadrilateral $ABCD$ is called cyclic if all its vertices are on a circle.

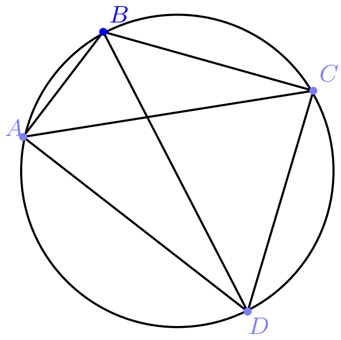
Lemma 6. ***(Isosceles trapezoid in a circle.)*** *Let $ABCD$ be a cyclic quadrilateral. The following are equivalent:*
a) $|AD| = |BC|$.
b) $\overset{\frown}{AD} = \overset{\frown}{BC}$.
c) $AB \parallel CD$.

Proof. a) $|AD| = |BC| \iff \triangle OAD \equiv \triangle OBC$ (case SSS) $\iff \widehat{AOD} = \widehat{BOC} \iff$
b) $\overset{\frown}{\mathrm{AD}} = \overset{\frown}{\mathrm{BC}} \iff \widehat{ABD} = \frac{\overset{\frown}{\mathrm{AD}}}{2} = \frac{\overset{\frown}{\mathrm{BC}}}{2} = \widehat{BDC}$ (angles on the circle) $\iff$
c) $AB \parallel CD$. $\square$

Lemma 7. ***(Rectangle in a circle.)*** *Let $ABCD$ be a quadrilateral whose vertices are on a circle $\mathcal{C}(O, R)$. The following are equivalent:*
a) $ABCD$ is a rectangle.
b) AC and BD are diameters of the circle (i.e., A, O, D are collinear, and B, O, C are collinear).

Proof: Exercise.

Theorem 8. *Let $ABCD$ be a quadrilateral. The following are equivalent:*
a) The quadrilateral $ABCD$ is cyclic.
b) $\widehat{ABD} = \widehat{ACD}$. (The angle formed by a diagonal with a side is equal with that formed by the other diagonal with the opposite side).
d) $\widehat{ABC} + \widehat{ADC} = 180°$ (The sum of two opposite angles is $180°$).

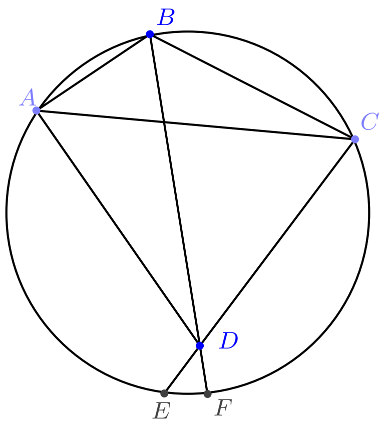


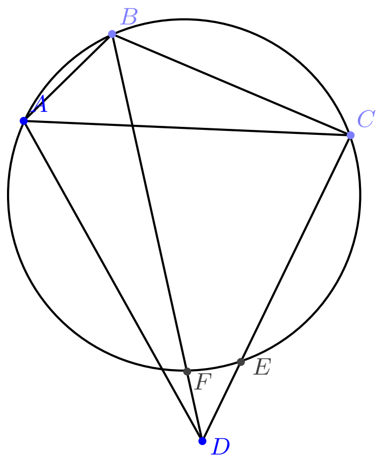
Proof. Part I: we assume a). We prove b) and c). Indeed, on the circle $\mathcal{C}$ containing the vertices A, B, C, D, we have

$$\widehat{ABD} = \widehat{ACD} = \frac{\overset{\frown}{\mathrm{AD}}}{2} \text{ and}$$
$$\widehat{ABC} + \widehat{ADC} = \frac{\overset{\frown}{\mathrm{ADC}} + \overset{\frown}{\mathrm{ABC}}}{2} = \frac{360^\circ}{2} = 180^\circ.$$

Here $\overset{\frown}{\mathrm{ADC}}$ denotes the arc bounded by A and C and containing the point D, while $\overset{\frown}{\mathrm{ABC}}$ denotes the arc bounded by A and C and containing the point B.

Part II: Assume b). Prove a). In this case, we let $\mathcal{C}$ be the circle containing the points A, B, C. This is the circle whose center is the circumcenter O of the triangle ABC (the intersection of the perpendicular bisectors), and whose radius is $|OA|$. We would like to prove that D is also a point on the circle $\mathcal{C}$. Proof by contradiction: assuming D is not on the circle $\mathcal{C}$, let $\mathcal{C}$ intersect the line CD at the point E, the line BD at the point F.





We have

$$\widehat{ABD} = \frac{\overset{\frown}{\mathrm{AF}}}{2} \text{ while } \widehat{ACD} = \frac{\overset{\frown}{\mathrm{AE}}}{2},$$

as angles on the circle. By assumption, we know $\widehat{ABD} = \widehat{ACD}$, and so by the equations above, $\overset{\frown}{\mathrm{AF}} = \overset{\frown}{\mathrm{AE}}$.

However, if D is outside the circle $\mathcal{C}$, then F is inside the arc $\overset{\frown}{\mathrm{AE}}$ and so $\overset{\frown}{\mathrm{AF}} < \overset{\frown}{\mathrm{AE}}$. Contradiction.

However, if D is inside the circle $\mathcal{C}$, then E is inside the arc $\overset{\frown}{\mathrm{AF}}$ and so $\overset{\frown}{\mathrm{AE}} < \overset{\frown}{\mathrm{AF}}$. Contradiction.

Part II: Assume c). Prove a). Similar with the previous part. We let $\mathcal{C}$ be the circle containing the points A, B, C. We would like to prove that D is also a point on the circle $\mathcal{C}$. Proof by contradiction: assuming D is not on the circle $\mathcal{C}$, let $\mathcal{C}$ intersect the line CD at the point E, the line AD at the point L. We have

$$\widehat{ABC} = \frac{\overset{\frown}{\mathrm{ALC}}}{2},$$

as angle on the circle, and

$$\widehat{ACD} = \frac{\overset{\frown}{\mathrm{ABC}} \pm \overset{\frown}{\mathrm{LE}}}{2},$$

as angle which is either internal, or external to the circle. By assumption, we know $\widehat{ABC} + \widehat{ADC} = 180°$, and so by the equations above,

$$\overset{\frown}{\mathrm{ALC}} + \overset{\frown}{\mathrm{ABC}} \pm \overset{\frown}{\mathrm{LE}} = 360°.$$

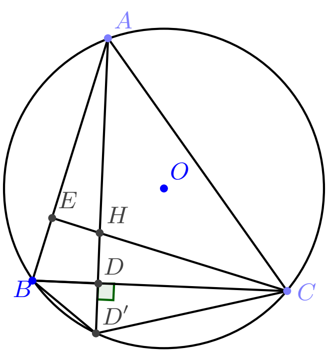
However,

$$\overset{\frown}{\mathrm{ALC}} + \overset{\frown}{\mathrm{ABC}} = 360°$$

as they span the entire circle, so it must be that $\overset{\frown}{\mathrm{LE}} = 0$, meaning that $L = E$. But this would mean that the lines CD and AD intersect the circle at the same point $E = L$. As the intersection of AD and CD is D, we must have $D = E = L$.

$\square$

Example 9. Let H be the orthocentre of $\triangle ABC$, and let D' denote the symmetric of H through BC. Then D' is a point on the circumcentre of $\triangle ABC$.



Proof. BC is the perpendicular bisector of $HD' \Longrightarrow \triangle CDH \equiv \triangle CDD'$. Then $\widehat{AD'C} = \widehat{DHC} = 90° - \widehat{HCD} = \widehat{ABC}$ so the quadrilateral $ABD'C$ is cyclic.
(We used $HD \perp BC$ and $CH \perp AB$.) □

1. Similar triangles

Definition 10. We say that $\triangle ABC$ and $\triangle A'B'C'$ are similar if their respective angles are equal: $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$, $\hat{C} = \hat{C}'$. we write

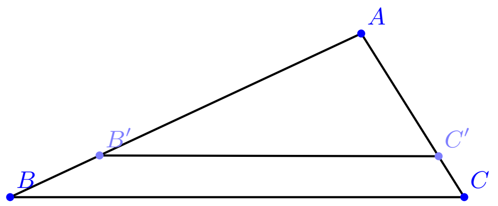
$$\triangle ABC \sim \triangle A'B'C'$$

Theorem 11. *If $\triangle ABC \sim \triangle A'B'C'$ then*

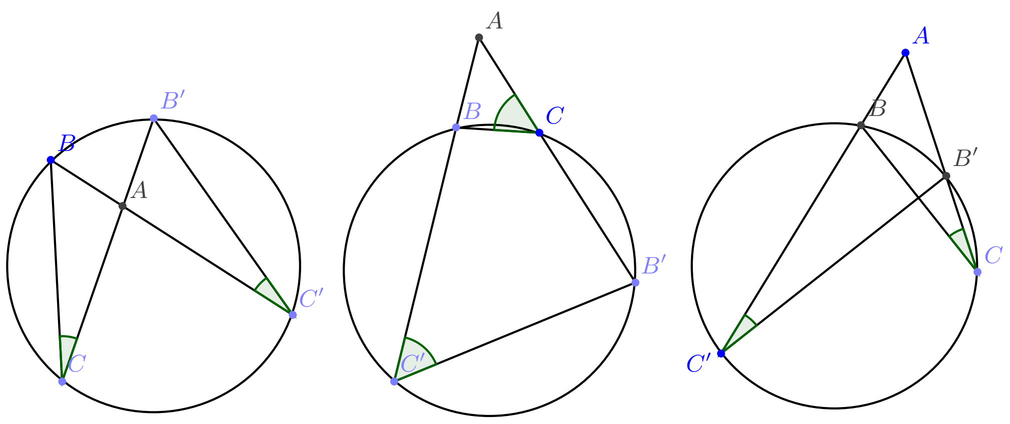
$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}.$$

The most common situations when two similar triangles arise are the following:

Theorem 12. ***Parallel lines:*** *$BC \parallel B'C' \Longrightarrow \triangle ABC \sim \triangle AB'C'$.*



Theorem 13. ***Anti-parallel lines:***
B, C, C', B' on the same circle and BC' intersects CB' at $A \Longrightarrow \triangle ABC \sim \triangle AB'C'$.



In this case, BC and $B'C'$ are called **anti-parallel**.

Please read the Notes on Areas and solve Ex. Set. 3 for more practice with similar triangles.

In particular, we can solve the following question: Given a circle $\mathcal{C}(O, R)$ and a point P, how can we describe how far the point is from the circle?

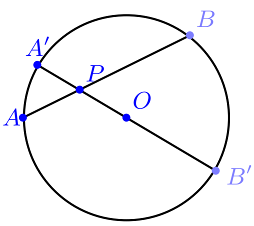
Definition 14. The power of a point P with respect to a circle is the product of the segments made by the point P on any chord passing through P.

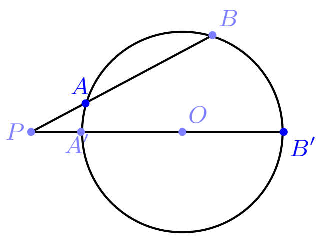
Question (9) from the Ex. Set. 3 can be reformulated as follows:

Theorem 15. *The **power of a point** P with respect to a circle $\mathcal{C}(O, R)$ does not depend on the chord on which it is calculated:*

$$|PA| \cdot |PB| = |PA'| \cdot |PB'| = \pm(|PO|^2 - R^2),$$

$+$ if P is outside the circle and $-$ if P is inside the circle.





Proof. Due to the equal angles on the circle,

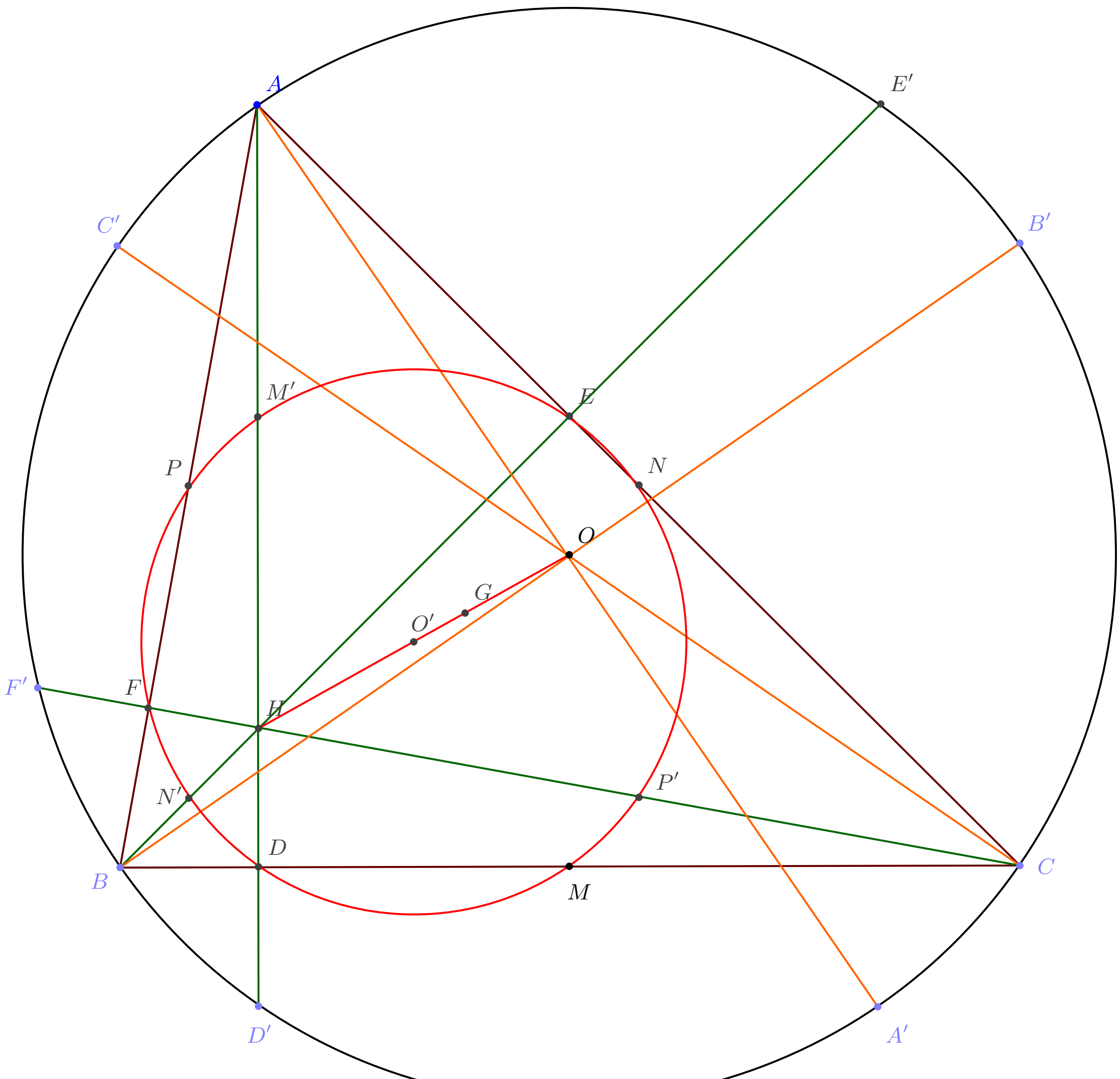
$$\triangle PAA' \sim \triangle PB'B \Longrightarrow \frac{|PA|}{|PB'|} = \frac{|PA'|}{|PB|}$$
$$\Longrightarrow |PA||PB| = |PA'||PB'| = \pm(|OP| - R)(|OP| + R).$$

□

Euler's diagram

One of the nicest structures in the geometry of a triangle is on the next page. Can you guess the significance of each point? If you were to connect all labeled points, could you find

- at least 19 segments whose midpoints are labeled in?
- at least 9 perpendicular bisectors?
- at least 15 parallelograms which are not rectangles?
- at least 9 isosceles trapezoids?
- at least 6 diameters and 9 rectangles?
- At least 9 pairs of similar triangles sharing H as common vertex? At least 12 pairs of similar triangles sharing A as common vertex?
- At least 8 triangles having G as centroid?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Useful Theorems

These are a few basic useful facts in Euclidean Geometry:

2. Euclid's 5 postulates :

Postulate (1) A unique straight line segment can be drawn joining any two distinct points.
Postulate (2) Any straight line segment is contained in a unique straight line.
Postulate (3) Given any straight line segment, a unique circle can be drawn having the segment as radius and one endpoint as center.
Postulate (4) All right angles are congruent.
Postulate (5) Given a point not on a given line, there exists a unique line through that point parallel to the given line.

Theorem 16. ***Parallel lines*** *Two distinct lines crossed by another line at the points P and Q are parallel $\Longleftrightarrow$ the following equivalent properties hold*

(A) *Two alternate angles at P and Q are equal;*
(B) *Two corresponding angles at P and Q are equal.*
(C) *Two interior consecutive angles add up to* $180°$

Theorem 17. ***Angles of a Triangle*** *The sum of all the interior angles of a triangle is* $180°$.

Theorem 18. ***Cases Sufficient for Congruent Triangles***

(A) *S.A.S. (Two pairs of sides and the angles between them are equal, respectively).*
(B) *A.A.S. (Two pairs of angles and a pair of sides are equal, respectively).*
(C) *S.S.S. (Each of the three pairs of sides are equal).*

Theorem 19. ***Isosceles Triangle*** *Consider a triangle ABC. The following two statements are equivalent:*

(A) *$\triangle ABC$ is isosceles with $|AB| = |AC|$.*
(B) *$\hat{B} = \hat{C}$.*

Theorem 20. ***Parallelogram*** *Let $ABCD$ be a quadrilateral. The following statements are equivalent:*
(A) $ABCD$ is a parallelogram.
(B) One pair of opposites sides are parallel and equal.
(C) The diagonals AC and BD intersect at their midpoint.
(D) Opposite sides are equal.
(E) Opposite angles are equal.

Theorem 21. ***Rectangle*** *Let $ABCD$ be a quadrilateral. The following statements are equivalent:*
(A) $ABCD$ is a rectangle (i.e. parallelogram with a right angle).
(B) The diagonals AC and BD intersect at their midpoint and $|AC| = |BD|$.

Theorem 22. ***Rhombus*** *Let $ABCD$ be a quadrilateral. Show that the following statements are equivalent:*
(A) $ABCD$ is a rhombus (i.e. parallelogram with equal sides).
(B) The diagonals AC and BD intersect at their midpoint and $AC \perp BD$.

Lemma 23. ***(Angle on a circle.)*** *Let A, B, C be three points on a circle. Let $\widehat{BC}$ denote the arc which does not contain A. Then*

$$\widehat{BAC} = \frac{\widehat{BOC}}{2} =: \frac{\widehat{BC}}{2}.$$

Theorem 24. ***Cyclic quadrilateral*** *Let $ABCD$ be a quadrilateral. The following are equivalent:*
(A) The quadrilateral $ABCD$ is cyclic.
(B) The angle formed by a diagonal with a side is equal with that formed by the other diagonal with the opposite side.
(C) The sum of two opposite angles is $180°$.

Theorem 25. ***The circumcircle of a triangle.*** *The perpendicular bisectors of the sides in a triangle ABC intersect at a point O, which is the centre of the unique circle containing the vertices of the triangle. The point O is called the* ***circumcentre*** *of the triangle, and the circle is called the* ***circumcircle*** *of the triangle.*

Example 26. The circumcentre of a right angled triangle is always the midpoint of the hypothenuse.

Proposition 27. ***Midline.*** *A midline in a triangle is parallel to, and equal to half of, the side opposite to it.*

Theorem 28. ***The centroid of a triangle.*** *The medians in a triangle ABC intersect at a point G, which forms triangles of equal areas with all the sides of the triangle. The point G is called the* ***centroid, or centre of gravity*** *of the triangle, and cuts each median in segments in the ratio $1:2$.*

Theorem 29. ***The incircle of a triangle.*** *The angle bisectors in a triangle ABC intersect at a point I, which is the centre of the unique circle inside the triangle touching the sides of the triangle. The point I is called the* ***incentre*** *of the triangle, and the circle is called the* ***incircle*** *of the triangle.*

Theorem 30. ***The orthocentre of a triangle.*** *The altitudes in a triangle ABC intersect at a point H called the* ***orthocentre*** *of the triangle.*