Cyclic Quadrilaterals - Angle Calculations

(1) Let ABC be a triangle. Consider three points A', B' and C' on BC, AC and AB, respectively. The circumcircles of $\triangle AB'C'$ and $CA'B'$ intersect at two points B' and S. Prove that $BA'SC'$ is a cyclic quadrilateral.

(2) i) Prove that a parallelogram is cyclic if and only if it is a rectangle. In this case, the circumcentre is the intersection of the diagonals of the rectangle.
ii) If two parallel lines cross a circle at four points, they form either a rectangle or an isoscelles trapezium (trapezoid).

(3) Let A, B, C be three points on the circle and let AM be a line tangent to the circle (touching the circle at exactly one point). if B and M are on two different sides of the line AC, prove that

$$\widehat{MAC} = \widehat{ABC}.$$

(4) Let ABC be a triangle. Consider two points B' and C' on AB and AC, respectively. Show that BC and $B'C'$ are parallel if and only if the circumcircles of ABC and $AB'C'$ are tangent to each other. [Hint: two circles are tangent to each other at A if they have a common tangent at A.]

(5) Let ABC be a triangle, let O be the centre of its circumcircle. Let R be the radius of its circumcircle and let AA' be a diameter of the circumcircle. Prove that $\widehat{ABC} = \widehat{AA'C}$ and hence prove that

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R \text{ hence also } R = \frac{abc}{4 \text{ Area } (\triangle ABC)}.$$

(6) Let $ABCD$ be a convex quadrilateral, let E be the point at the intersection of the lines AD and BC and F the point at the intersection of the lines AB and CD.
a) Show that the circumcircles of the triangles EDC, EAB, FBC and FAD all intersect at a point M (Monge's point).
b) Prove that M is on the same circle with all the circumcentres of the circles above.

(7) * All the Monge's points for the quadrilateral in a pentagonal star are on a circle.

(8) a) Let H be the orthocentre of $\triangle ABC$. Let $AH \bigcap BC = \{D\}$, $BH \bigcap AC = \{E\}$, $CH \bigcap AB = \{F\}$.
a) Prove that $AEHF$ and $BCEF$ are cyclic quadrilaterals. Find four other cyclic quadrilaterals in the diagram.
b) Find all possible pairs of similar triangles in diagram.
c) Let M, N, P be the midpoints of the sides BC, CA and AB respectively, and M', N', P' the midpoints of AH, BH and CH. Prove that $MNM'N'$ is a rectangle. Find two other rectangles in the picture. Prove that $DMNP$ is an isosceles trapezoid. Find as many isosceles trapezoids in the picture as possible.
d) Prove that M, N, P together with D, E, F, and M', N', P' all lie on the same circle. This is called the 9-point circle, or Euler's circle.

(9) (Internal and external angles) Let $\mathcal{C}$ be a circle of center O and P a point in the interior of $\mathcal{C}$. Let A, B, C, D be points on $\mathcal{C}$ such that P is inside the segments AB and CD. Let Q be the intersection point of the lines AD and BC. Prove that

$$\widehat{APC} = \frac{1}{2}(\widehat{AOC} + \widehat{BOD}) \text{ and } \widehat{AQC} = \pm\frac{1}{2}(\widehat{AOC} - \widehat{BOD}).$$

(10) Let M be a point on the circumcircle of $\triangle ABC$ and let N, P, Q be points on the sides AB, BC, AC respectively such that $MN \perp AB$, as well as $MP \perp BC$ and $MQ \perp AC$. Prove that N, P, Q are collinear.

(11) Let $BCED$ be a cyclic quadrilateral whose diagonals EB and CD intersect at a point A. Let O be the circumcentre of $\triangle ABC$. Prove that $OA \perp DE$.

(12) * Let ABC be a triangle, let I be the incentre, let I_A, I_B and I_C be the centers of the circles externally tangent to sides. Let A', B', C' be the feet of the perpendiculars from I to the sides of the triangle, and A'', B'', C'' the feet of the perpendiculars from I_A to BC, from I_B to AC and from I_C to AB, respectively.

a) Show that I_AA, IBB, I_CC are the altitudes of $I_AI_BI_C$.

b) Show that I_AA'', I_BB'', I_CC'' intersect at the circumcircle O'' of $I_AI_BI_C$, of radius $O''I_A = 2R$, where R is the circumradius of ABC. (Hint: use 8)

c) Find all pairs of similar triangles in the figure. In particular, prove that

$$\frac{A'B'}{I_AI_B} = \frac{B'C'}{I_BI_C} = \frac{C'A'}{I_CI_A} = \frac{r}{2R},$$

where r is the inradius of ABC.

d) Show that I_AA', I_BB', I_CC' intersect at a point S. Furthermore, S is collinear with I, O'', the orthocentre of $A'B'C'$ and the centroids of $I_AI_BI_B$ and $A'B'C'$.

e) Let M, N, P be the midpoints of the sides BC, CA and AB respectively. The perpendicular from M to I_BI_C, from N to I_CI_A and from P to I_AI_B intersect at a point Q, such that Q, I and G are collinear, where G is the centroid of ABC, and $IG = 2QG$.

Anca Mustata, School of Mathematical Sciences, UCC

E-mail address: a.mustata@ucc.ie