

# Induction in Pascal's Triangle

Pascal's Formula, The Hockey Stick,  
The Binomial Formula, Sums

# Pascal's Triangle: click to see [movie](#)

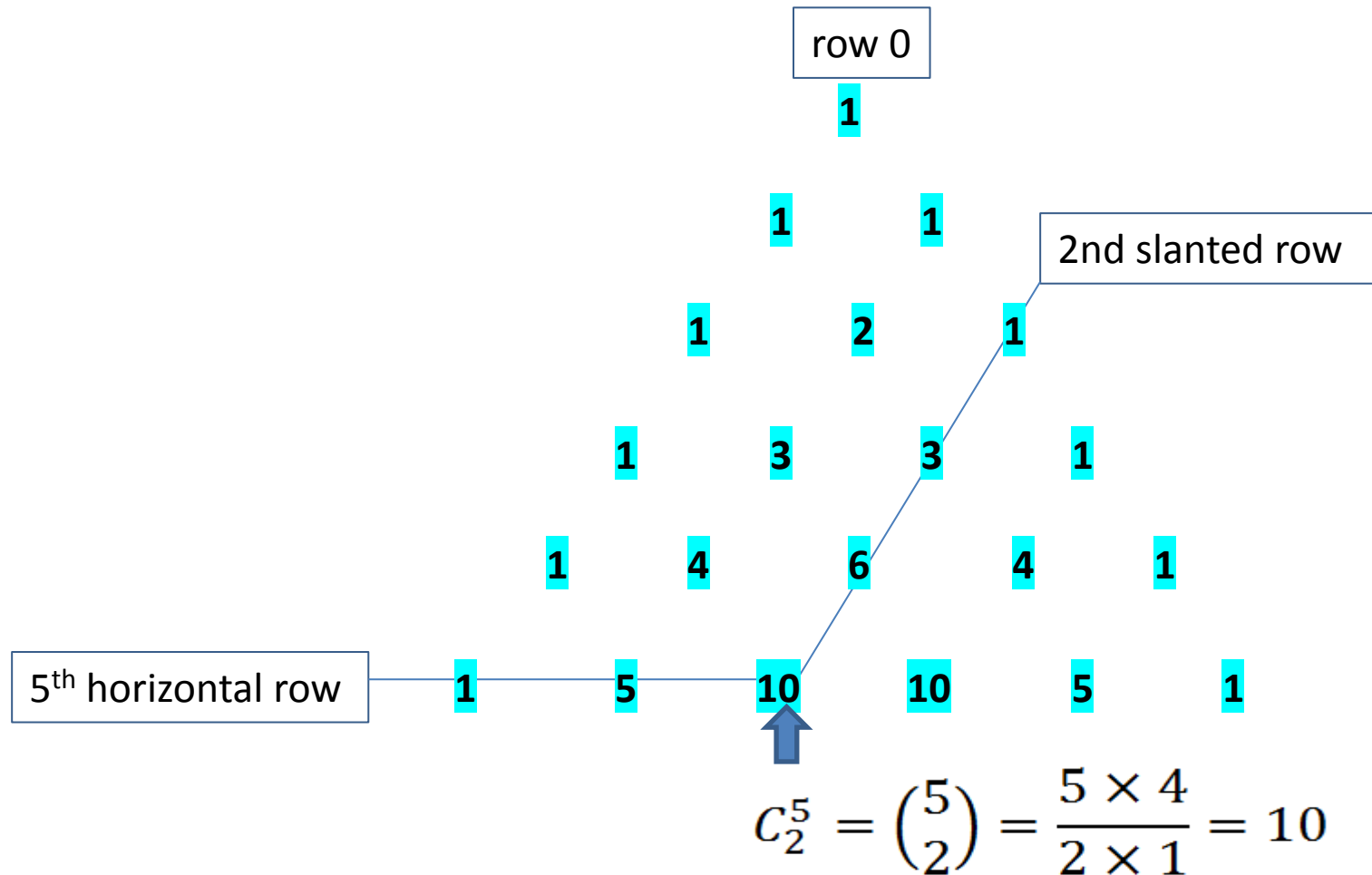
1										
1					1					
1				2	1					
1			3	3	1					
1		4	6	4	1					
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

## Count the rows in Pascal's triangle starting from 0.

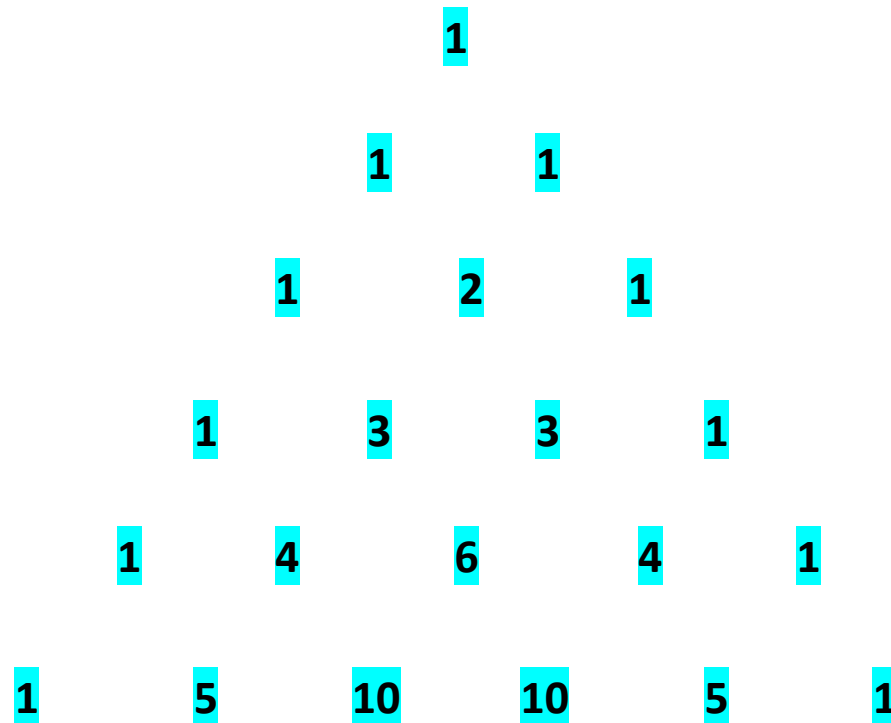
The entry on the  $n$ -th horizontal row, and  $k$ -th slanted row in Pascal's triangle:

$$\binom{n}{k} = C_k^n = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}$$

= number of ways to choose a set of  $k$  objects from among  $n$  given objects.



Write the suitable  $C_k^n$  or  $\binom{n}{k}$  notations besides each entry in Pascal's triangle



$$\mathbf{1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \mathbf{3} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \mathbf{3} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \mathbf{4} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \mathbf{6} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \mathbf{4} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \mathbf{5} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \mathbf{10} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \mathbf{10} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \mathbf{5} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \mathbf{1} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Practice with the formula  $\binom{n}{k} = C_k^n = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}$

$$C_3^6 = \binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

$$C_4^7 = \binom{7}{4} = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35.$$

Useful note: there are as many factors above as under the quotient line.

Calculate these numbers using the formula, then locate them in the triangle.

$$C_2^5 = \binom{5}{2} =$$

$$C_3^7 = \binom{7}{3} =$$

$$C_4^9 = \binom{9}{4} =$$

$$C_2^6 = \binom{6}{2} =$$

$$C_3^8 = \binom{8}{3} =$$

$$C_5^9 = \binom{9}{5} =$$

$$C_2^7 = \binom{7}{2} =$$

$$C_3^9 = \binom{9}{3} =$$

$$C_6^9 = \binom{9}{6} =$$

$$C_2^5 = \binom{5}{2} = \frac{5 \times 4}{2 \times 1} = 10$$

$$C_2^6 = \binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15$$

$$C_2^7 = \binom{7}{2} = \frac{7 \times 6}{2 \times 1} = 21$$

$$C_3^7 = \binom{7}{3} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

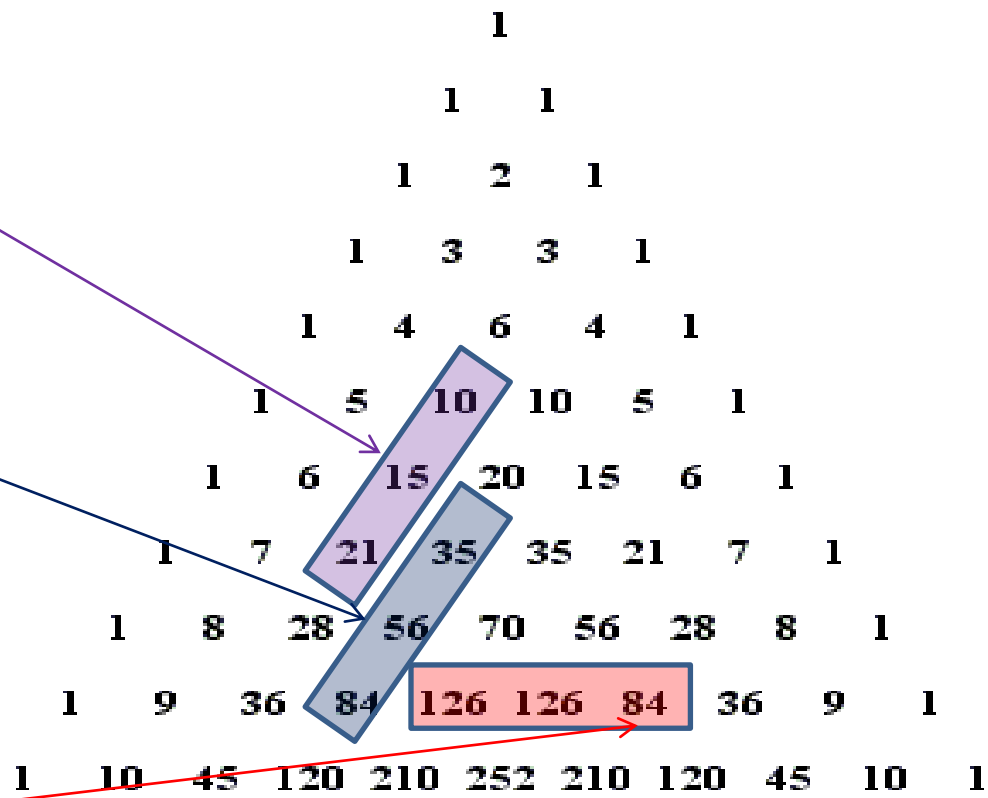
$$C_3^8 = \binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

$$C_3^9 = \binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

$$C_4^9 = \binom{9}{4} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$$

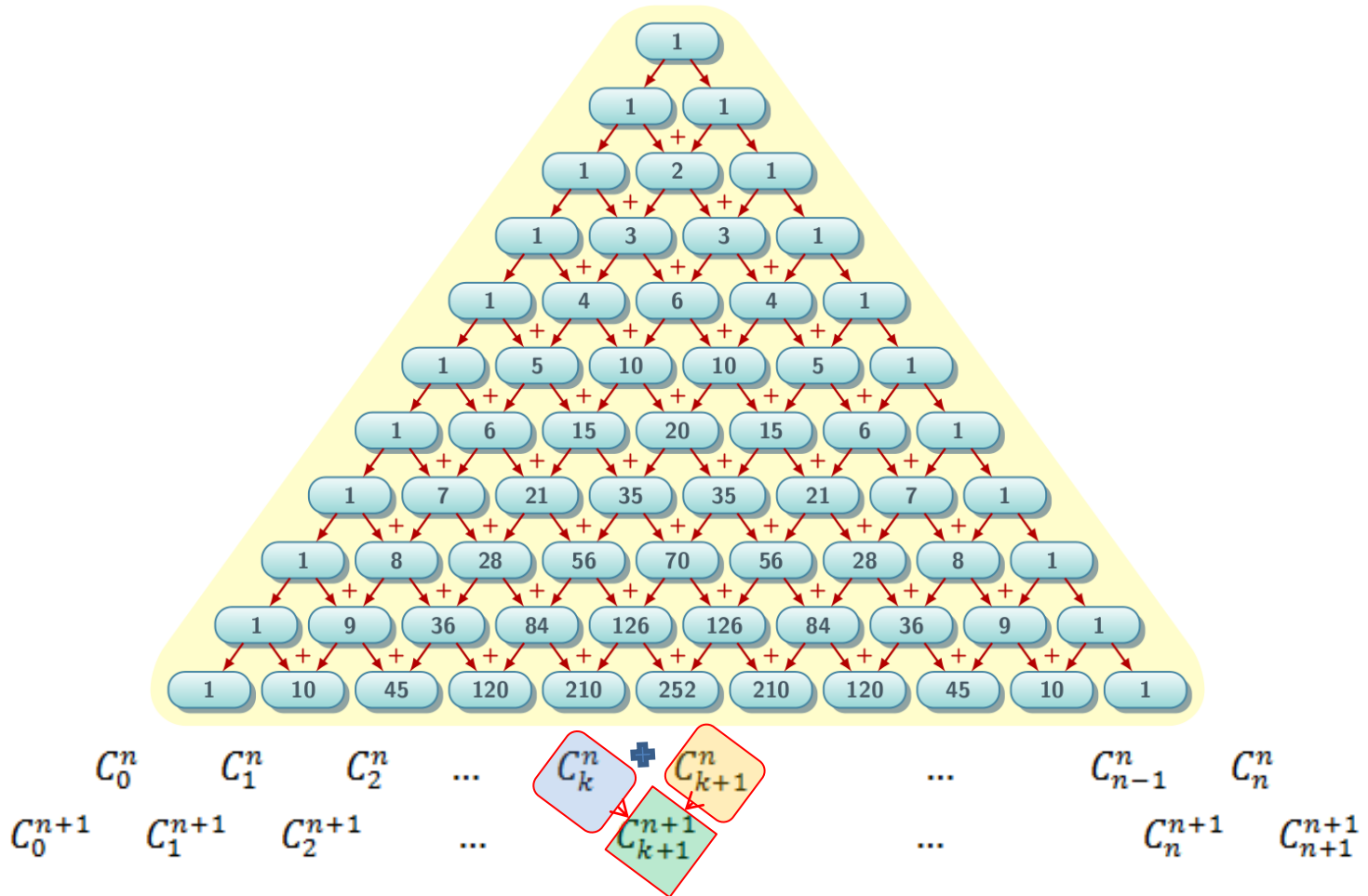
$$C_5^9 = \binom{9}{5} = \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1} = 126$$

$$C_6^9 = \binom{9}{6} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 84$$



Recall: Pascal's formula:

$$C_{k+1}^{n+1} = C_{k+1}^n + C_k^n$$





# Pascal's formula

## Combinatorial proof:

Suppose there are  $n+1$  people in your class. In how many ways can a group of  $k+1$  people be chosen?

$$C_{k+1}^{n+1} = C_{k+1}^n + C_k^n$$

Ways to choose  
a random group  
of  $k+1$  people

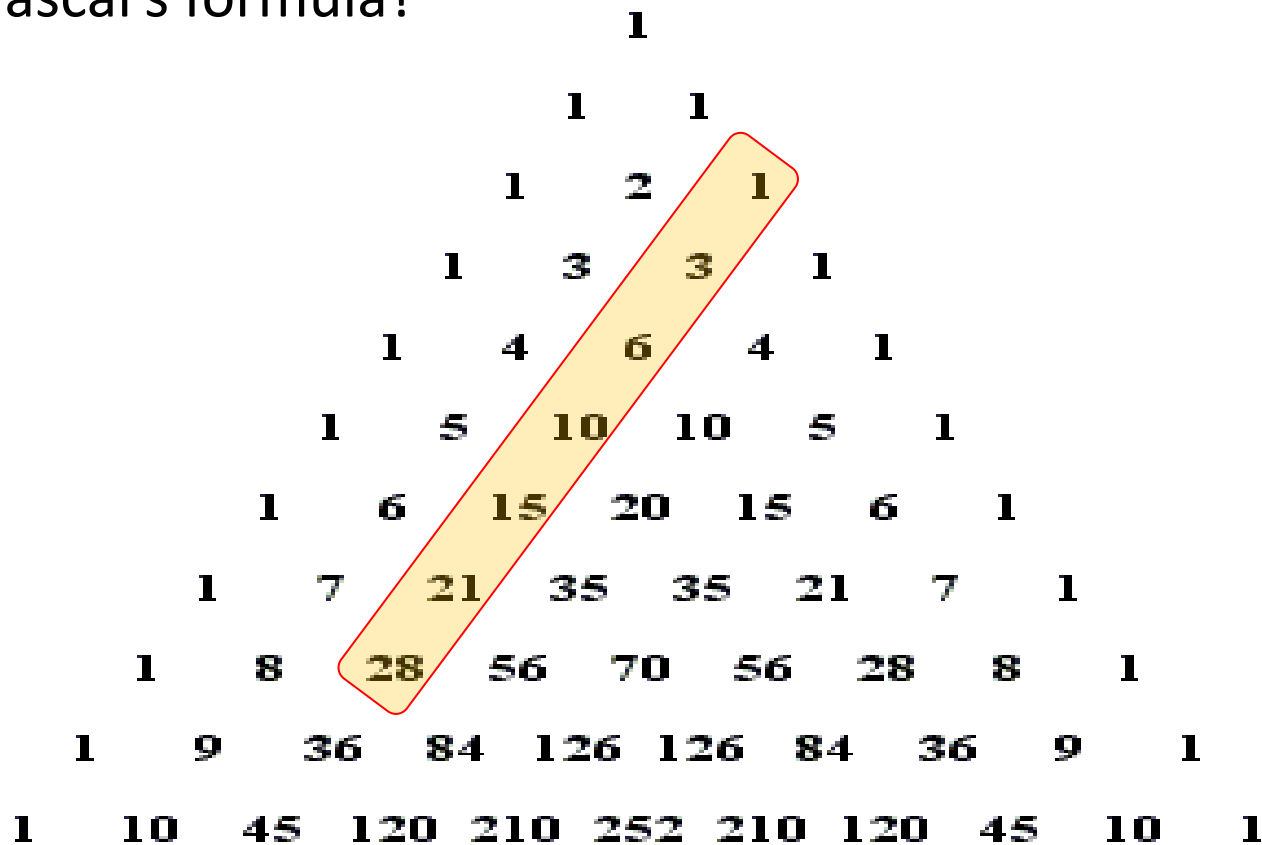
Ways to choose  
a group of  $k+1$   
people if you're  
not included

Ways to choose  
a group of  $k+1$  people  
if you're included  
(it remains to choose  $k$   
other people from among  
your  $n$  classmates)

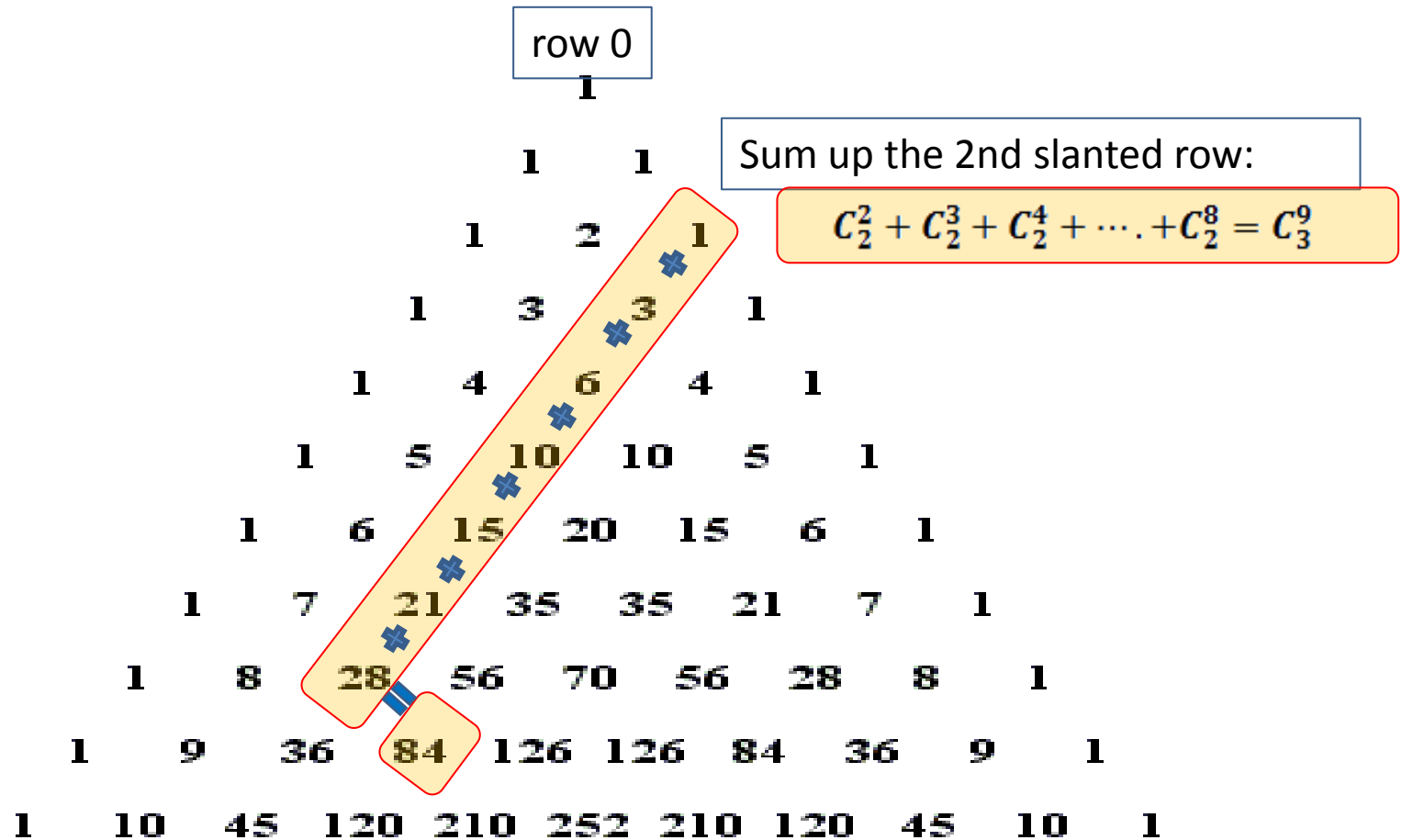
# The Hockey Stick Formula

Write down each of the numbers in the yellow line as  $C_k^n = \binom{n}{k}$

What can you say about the sum? Can you explain your result based on Pascal's formula?



# The Hockey Stick Formula

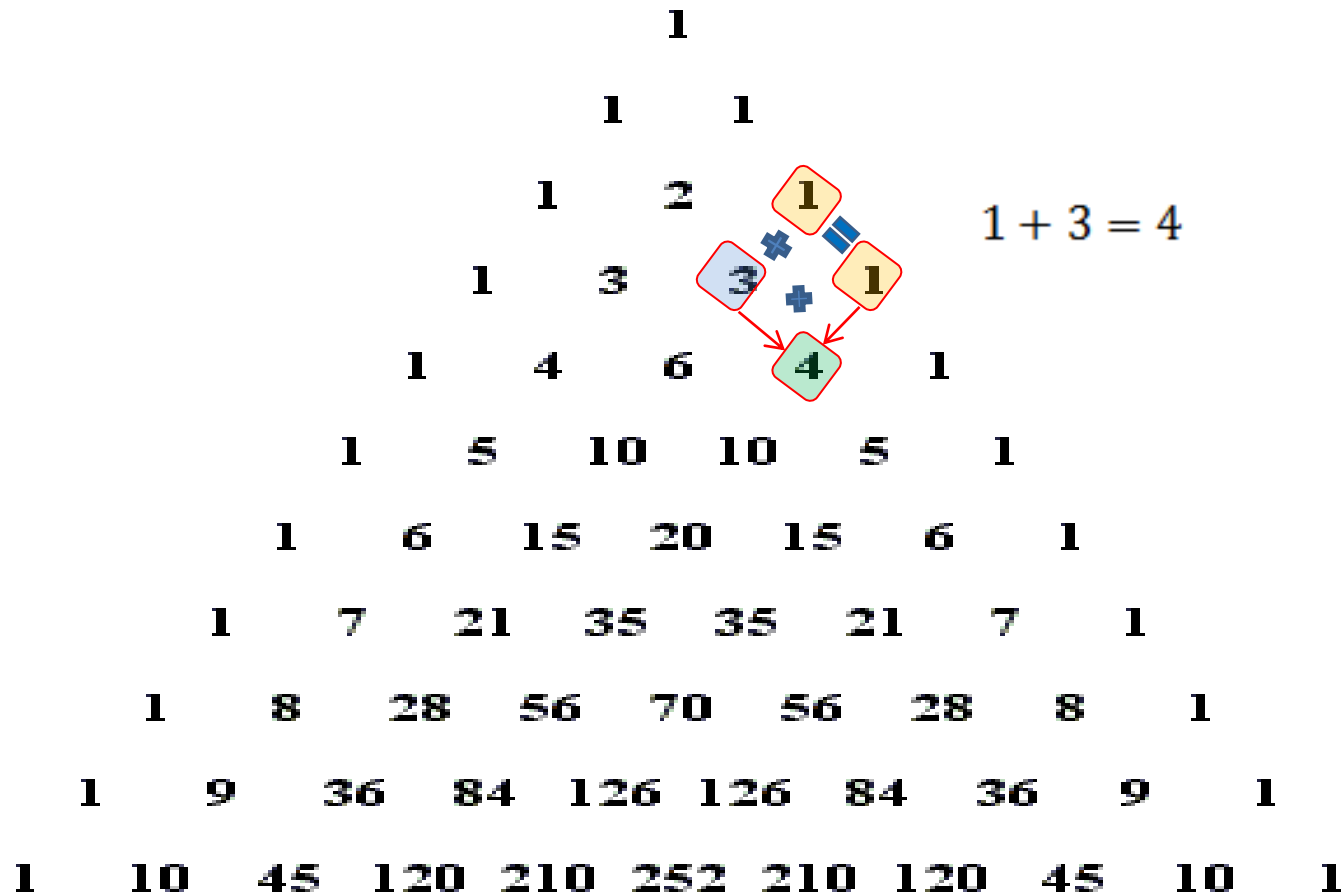


$$C_2^2 + C_2^3 + C_2^4 + \dots + C_2^n = C_3^{n+1}$$
$$C_2^2 + C_2^3 + C_2^4 + \dots + C_2^n = C_3^{n+1}$$

small

larger

step by step:

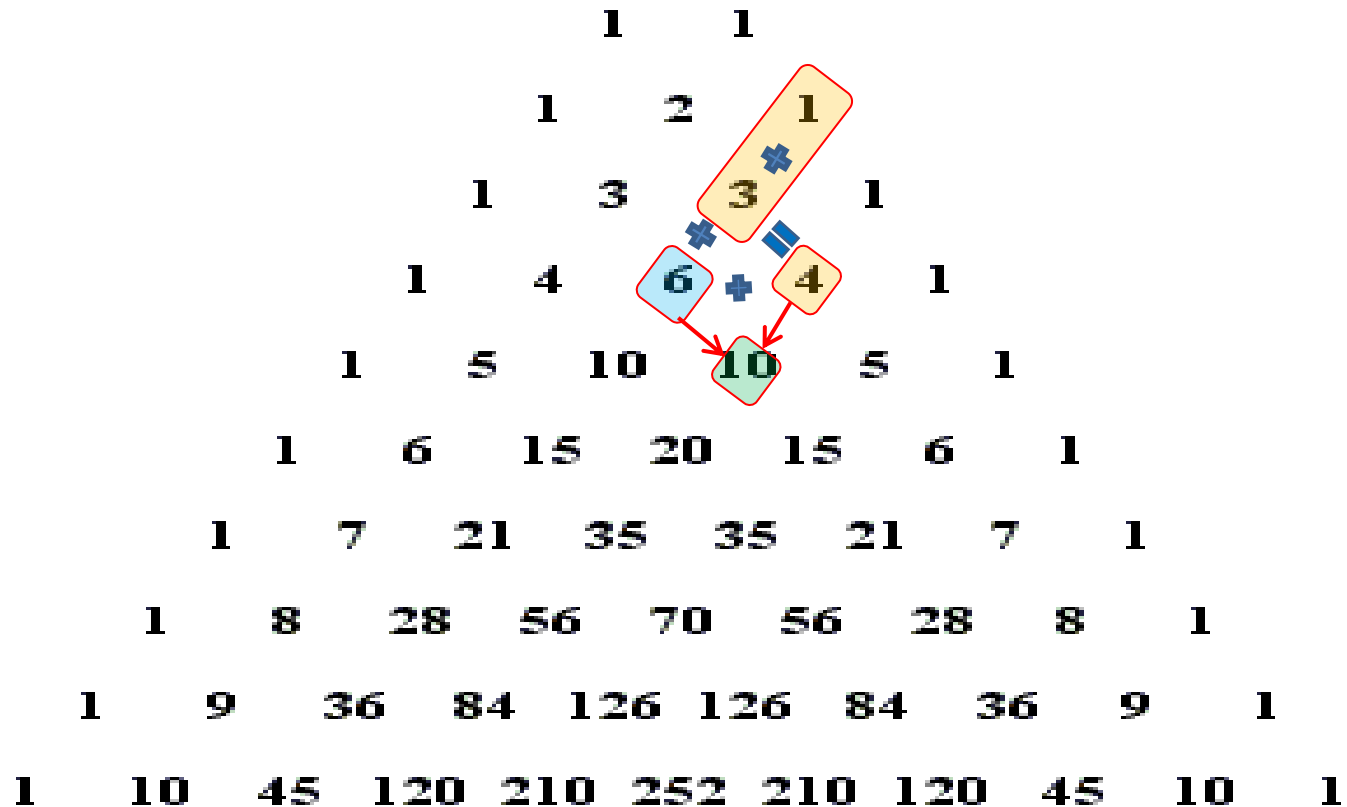


# The Hockey Stick Formula

Step 1:

$$(1+3)+6=4+6=10$$

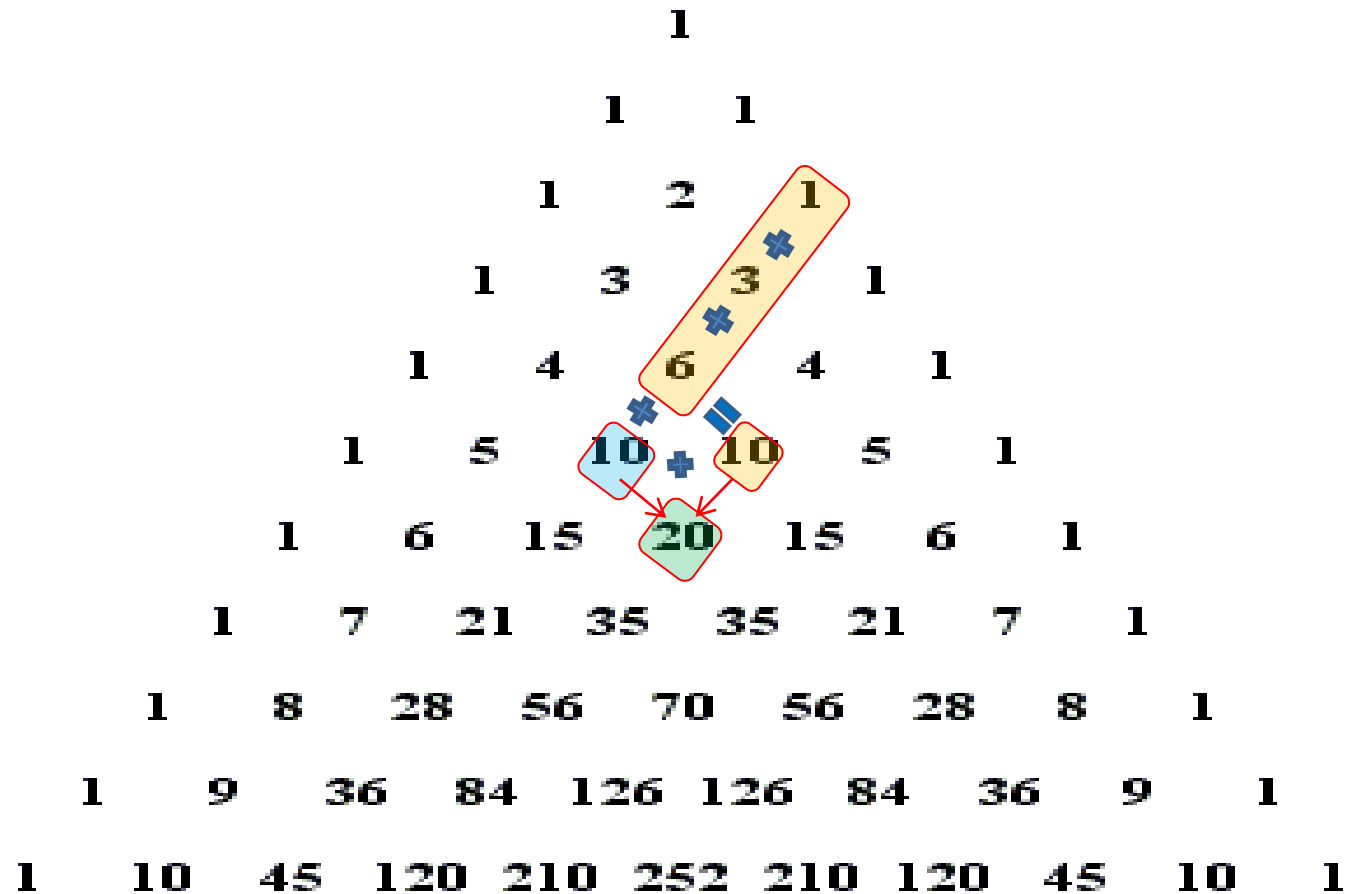
**1**



# The Hockey Stick Formula

Step 2:

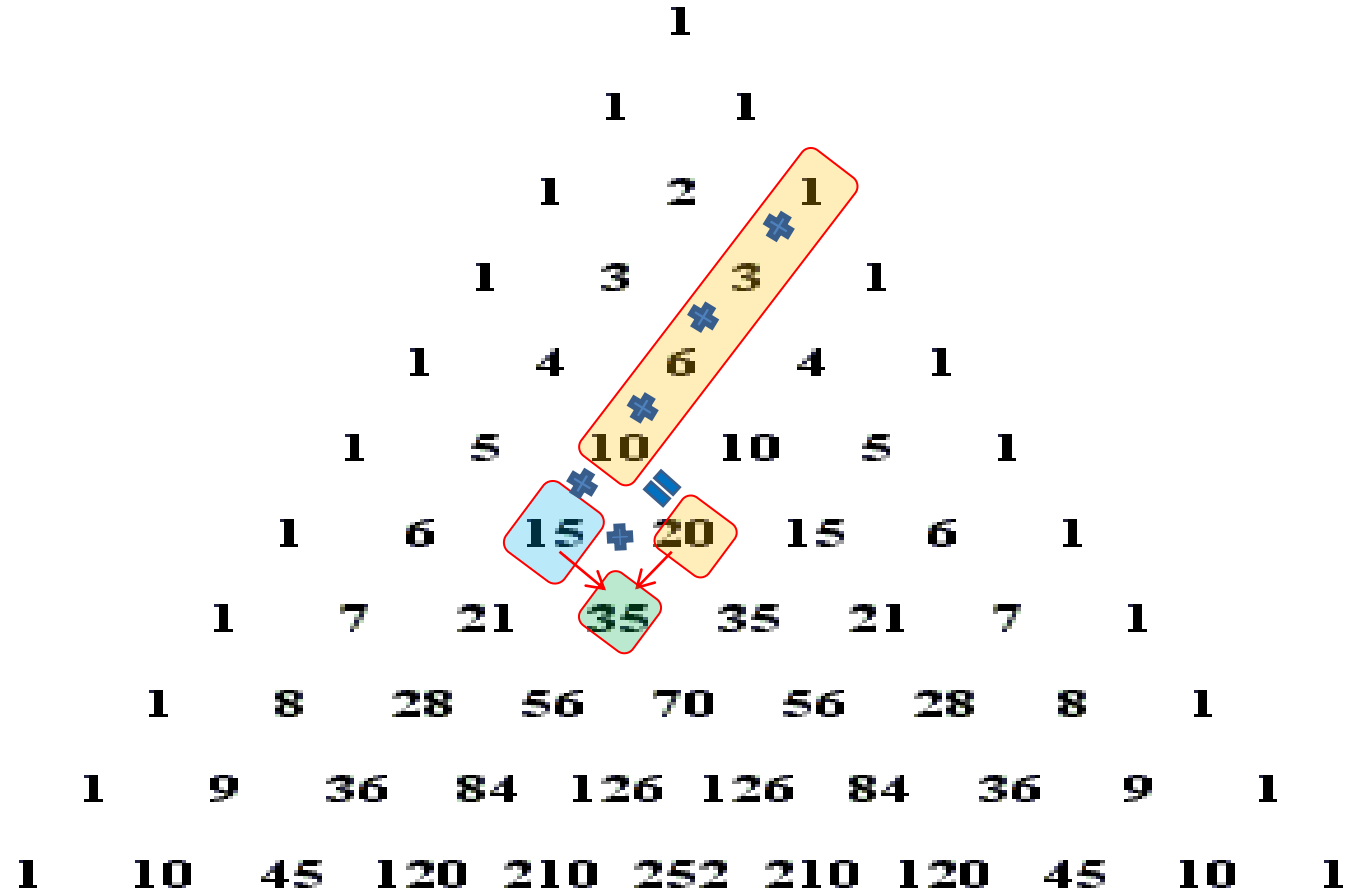
$$(1+3+6)+10=10+10=20$$



# The Hockey Stick Formula

### Step 3:

$$(1+3+6+10)+15=20+15=35$$



# 1, 2, skip a few, ..., $n$ , $n+1$

After a while it gets boring...

So let's skip some unknown number of steps, to

$$C_2^2 + C_2^3 + C_2^4 + \dots + C_2^n = C_3^{n+1} \quad \text{for some } n.$$

Presumably if we were patient enough, we could have reached this step by step, so **we assume that the formula is true for this number  $n$ .**

**Can we prove the formula at the next step, when  $n$  gets replaced by  $n+1$ ?**

If we can do this for any random  $n$ , then the formula is always true!

... Because we can reach any natural number, no matter how large, by a sequence of consecutive steps.



# The Hockey Stick Formula: Induction step:

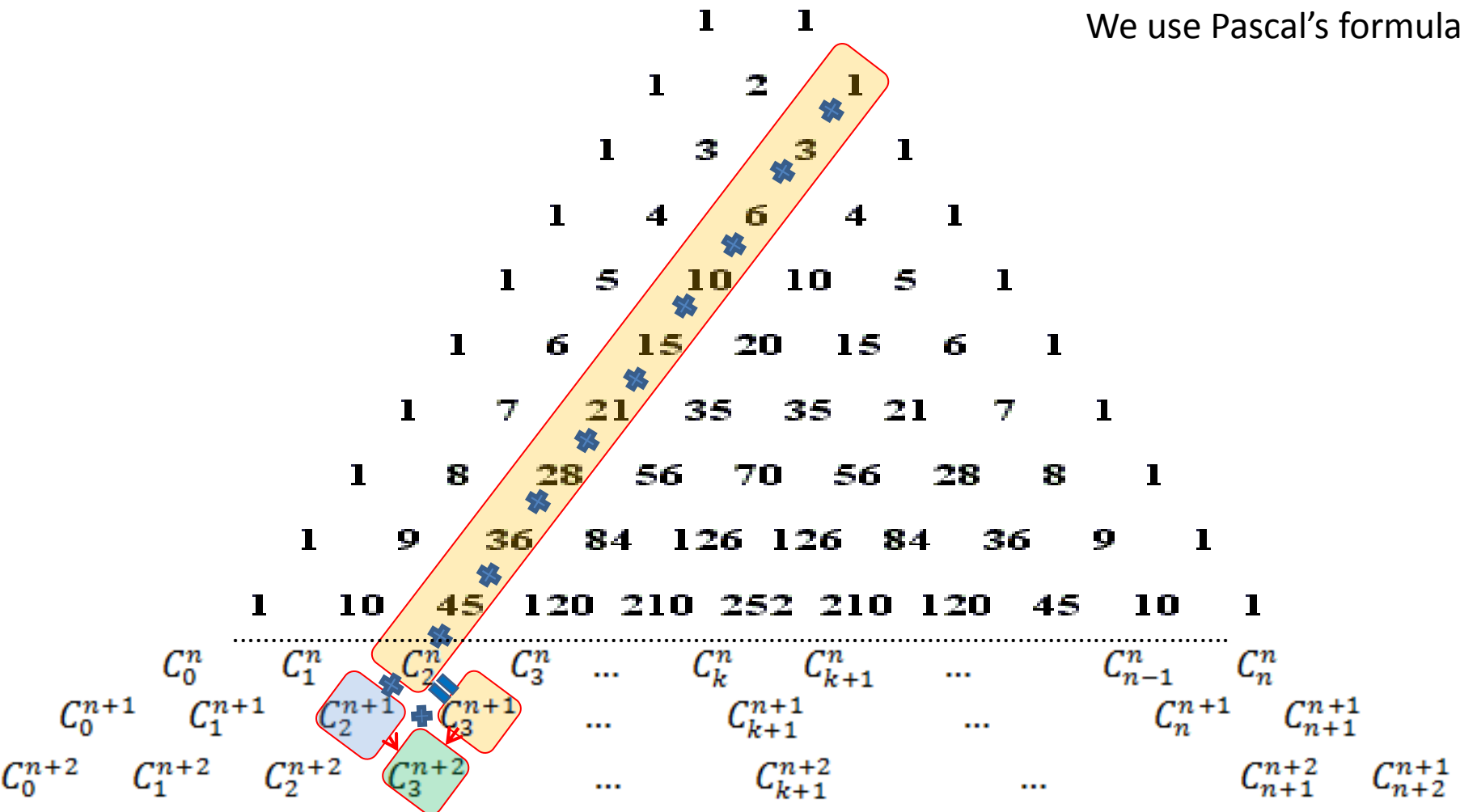
If

$$1 + 3 + 6 + \dots + C_2^n = C_3^{n+1}$$

Then

$$(1 + 3 + 6 + \dots + C_2^n) + C_2^{n+1} = C_3^{n+1} + C_2^{n+1} = C_3^{n+2}$$

We use Pascal's formula here.



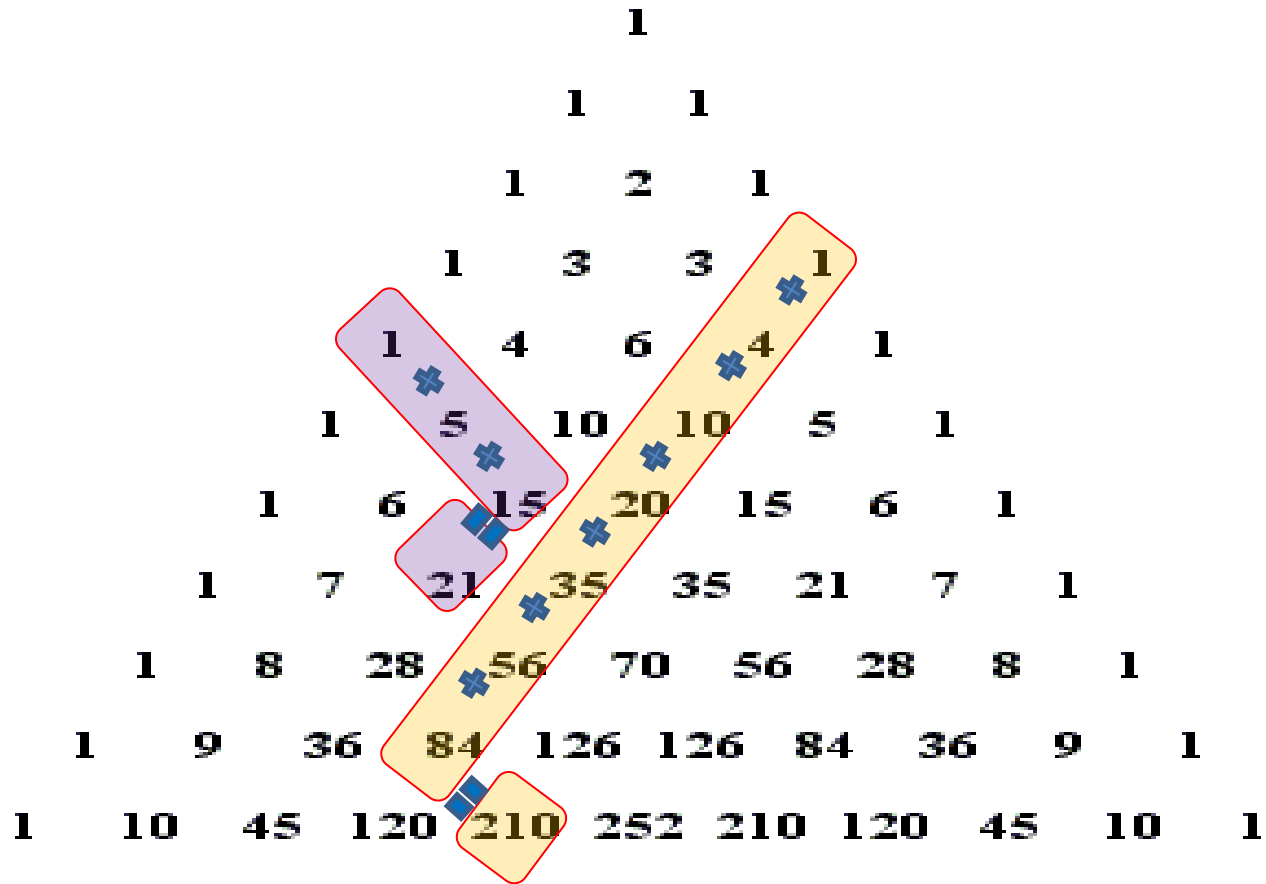
# Recap Induction:

- We checked the first few steps till we saw a pattern.
- We assumed the formula true for  $n$ .
- We proved the formula for  $n+1$ .

Once we understood the pattern, the last step was just a formality: All we had to do was figure out the correct notations. However, this step is necessary in order to convince ourselves (and others) that the pattern works forever, that it doesn't break down after finitely many steps.

# The Hockey Stick Formula

Formulate other Hockey Stick type rules. Prove them by Induction.



# The Binomial Formula

The entries in the n-th row of Pascal's Triangle are exactly the numbers coming up in the long form of  $(x + y)^n$

$$(x + y)^0 =$$

1

$$(x + y)^1 =$$

1x + 1y

$$(x + y)^2 =$$

1x<sup>2</sup> + 2x<sup>1</sup>y<sup>1</sup> + 1y<sup>2</sup>

$$(x + y)^3 =$$

1x<sup>3</sup> + 3x<sup>2</sup>y<sup>1</sup> + 3x<sup>1</sup>y<sup>2</sup> + 1y<sup>3</sup>

$$(x + y)^4 =$$

1x<sup>4</sup> + 4x<sup>3</sup>y<sup>1</sup> + 6x<sup>2</sup>y<sup>2</sup> + 4x<sup>1</sup>y<sup>3</sup> + 1y<sup>4</sup>

$$(x + y)^5 = 1x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5$$

Does this pattern hold forever and why?

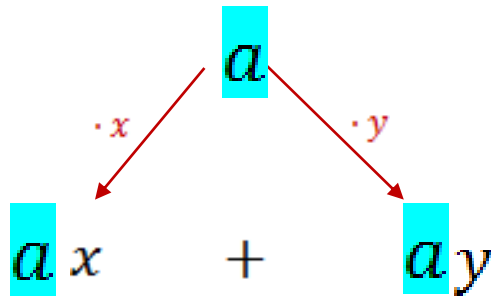
# The Binomial Formula

$$(x + y)^n = C_0^n x^n + C_1^n x^{n-1} y + C_2^n x^{n-2} y^2 + \cdots + C_k^n x^{n-k} y^k + \cdots + C_n^n y^n$$

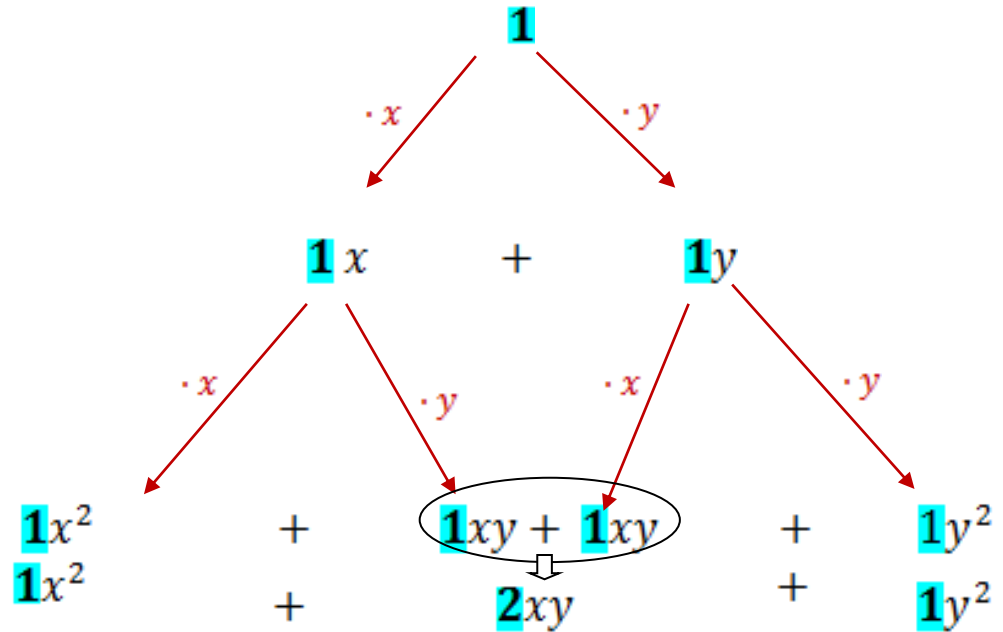
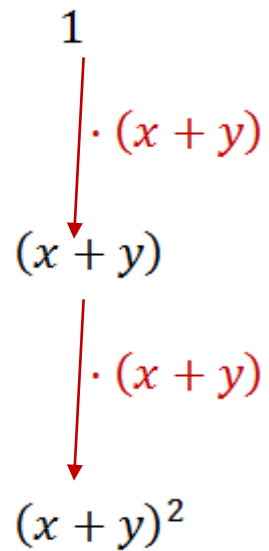
Proof: by Induction.

Main tool: Distributivity:

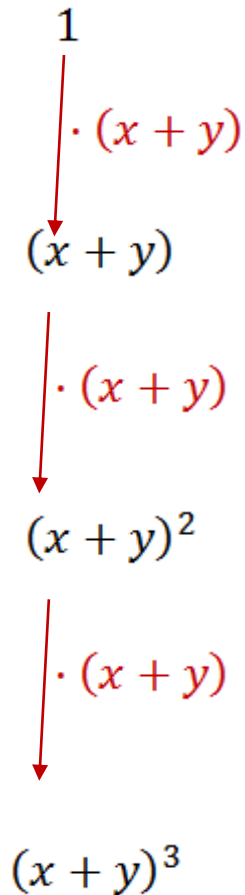
$$a(x + y) = ax + ay$$



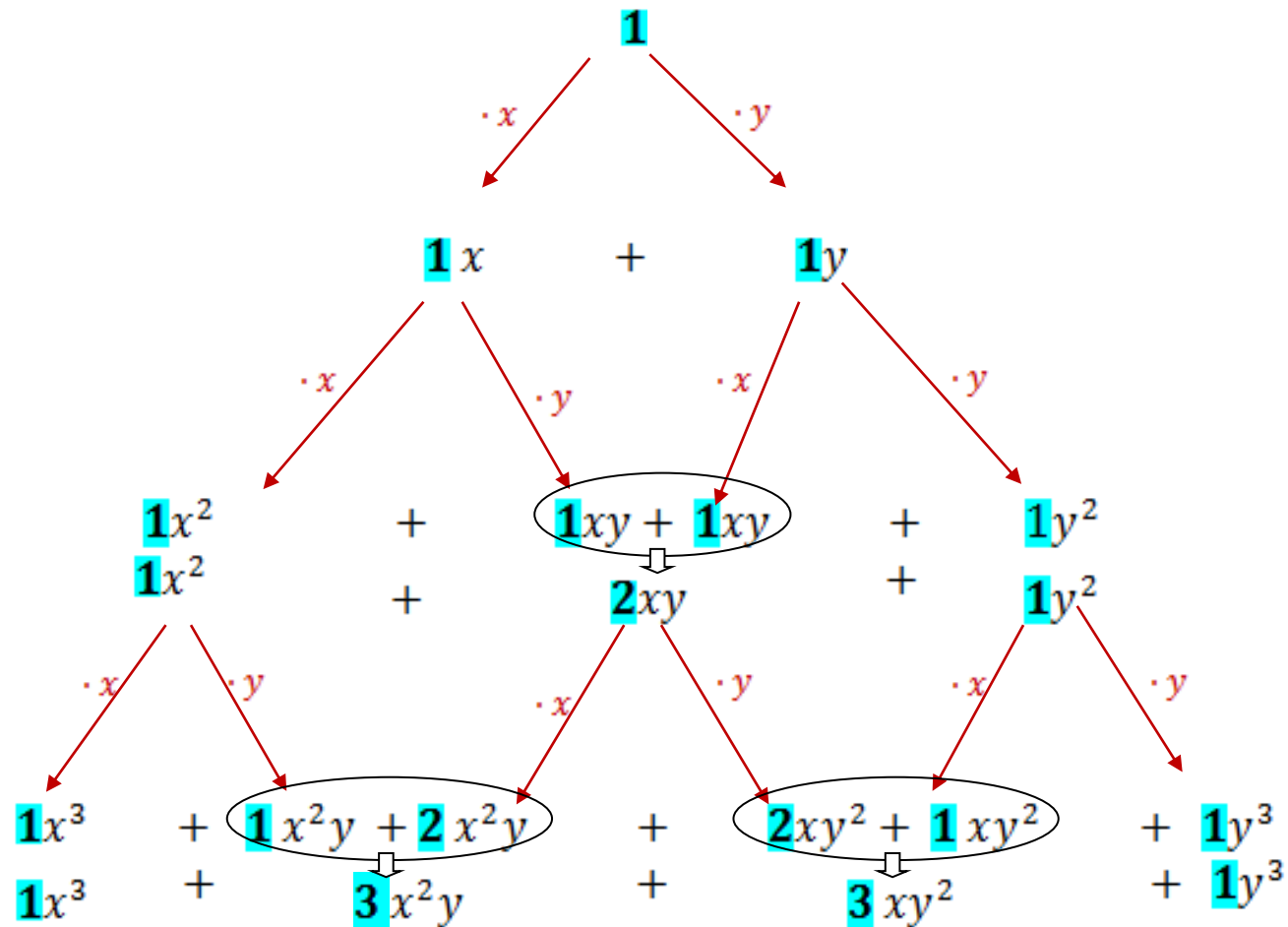
# The expansion of $(x + y)^2$



# The expansion of $(x + y)^3$



Short form



Long form

The highlighted numbers are generated by the same rule as in Pascal's triangle.

$$(x+y)^0 =$$

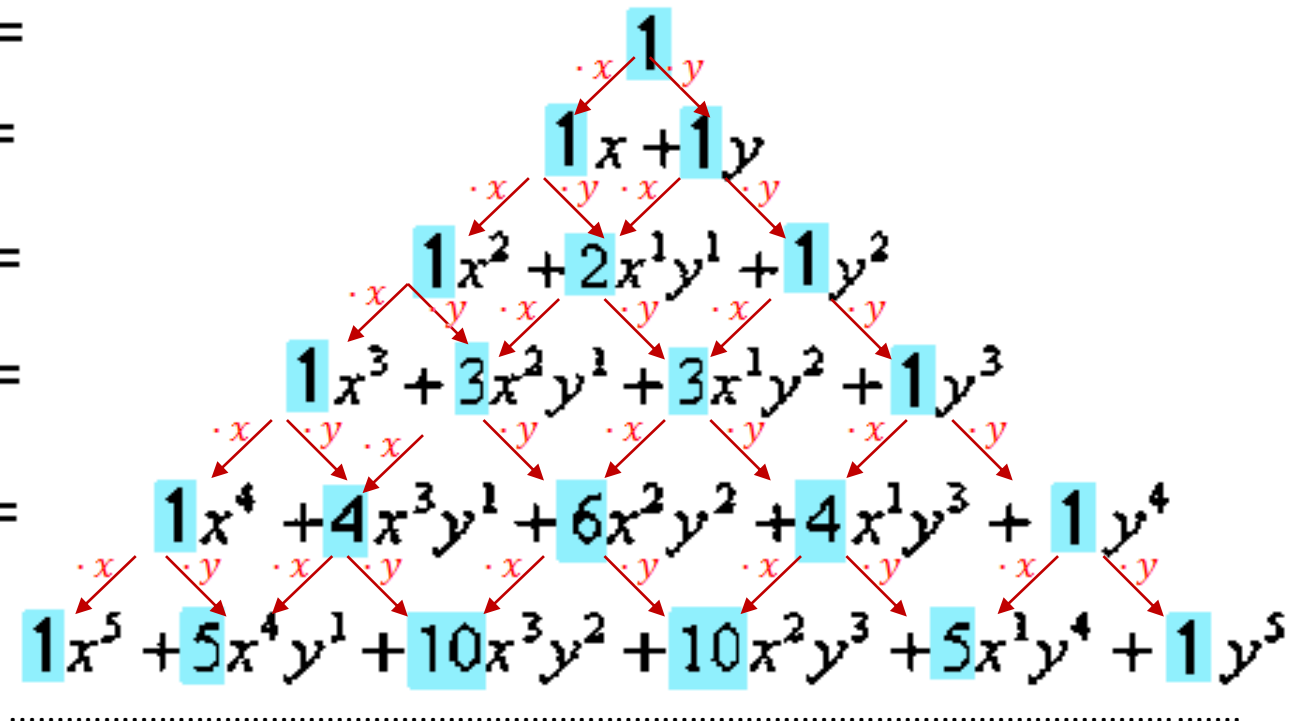
$$(x+y)^1 =$$

$$(x+y)^2 =$$

$$(x+y)^3 =$$

$$(x+y)^4 =$$

$$(x+y)^5 =$$



Assuming:

$$(x+y)^n = C_0^n x^n + C_1^n x^{n-1}y^1 + C_2^n x^{n-2}y^2 + \dots + C_k^n x^{n-k}y^k + C_{k+1}^n x^{n-k-1}y^{k+1} + \dots + C_n^n y^n$$

We multiply the last row by  $(x+y)$  so as to get  $(x+y)^{n+1}$



The highlighted numbers are generated by the same rule as in Pascal's triangle.

$$(x+y)^0 =$$

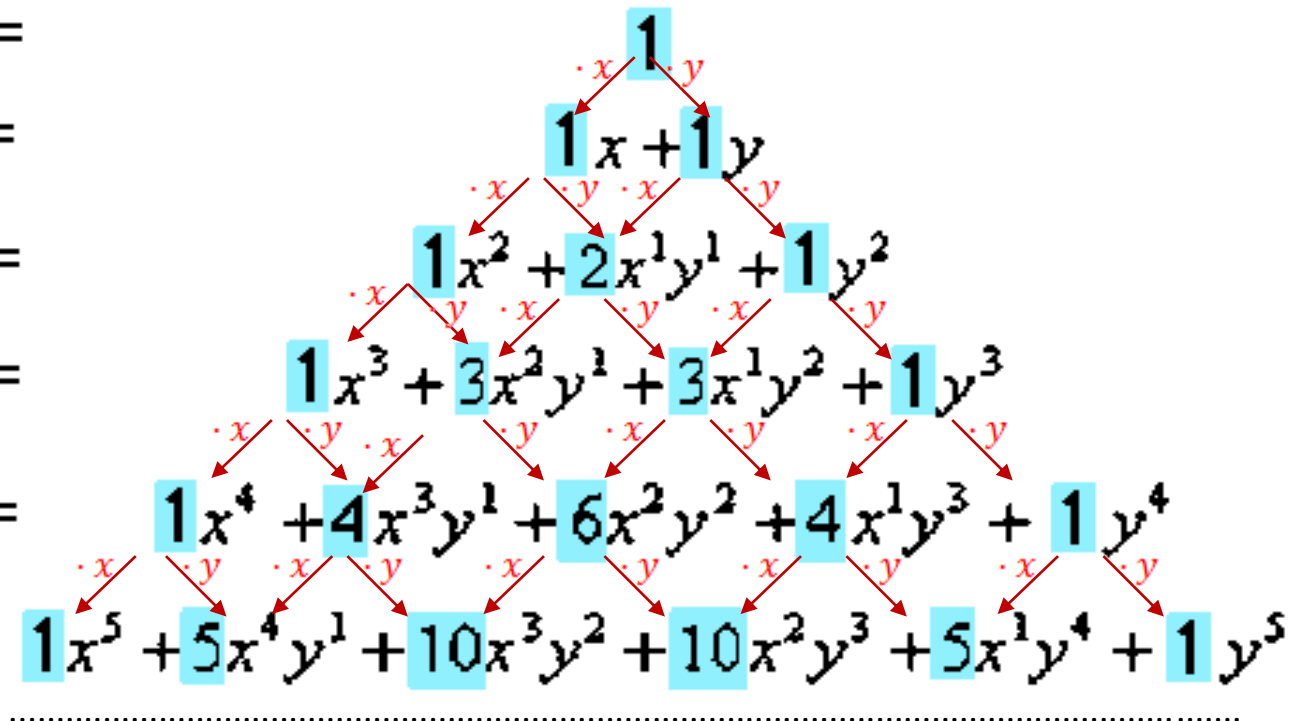
$$(x+y)^1 =$$

$$(x+y)^2 =$$

$$(x+y)^3 =$$

$$(x+y)^4 =$$

$$(x+y)^5 =$$



Assuming:

$$(x+y)^n = C_0^n x^n + C_1^n x^{n-1} y^1 + C_2^n x^{n-2} y^2 + \dots + C_k^n x^{n-k} y^k + C_{k+1}^n x^{n-k-1} y^{k+1} + \dots + C_n^n y^n$$

$$C_0^{n+1} x^{n+1} + C_1^{n+1} x^n y^1 + C_2^{n+1} x^{n-1} y^2 + \dots + C_{k+1}^{n+1} x^{n-k} y^{k+1} + \dots + C_{n+1}^{n+1} y^{n+1}$$

We use Pascal's formula  $C_{k+1}^{n+1} = C_{k+1}^n + C_k^n$  for each  $k$ .

We obtain the entries in the  $(n+1)$ -th row of Pascal's triangle.

We thus proved that the binomial formula also works for  $(x+y)^{n+1}$

# The Binomial Formula

$$(x + y)^n = C_0^n x^n + C_1^n x^{n-1} y + C_2^n x^{n-2} y^2 + \cdots + C_k^n x^{n-k} y^k + \cdots + C_n^n y^n$$

Combinatorial Proof:

$$\begin{aligned} (x + y)^n &= (x + y)(x + y) \cdots (x + y) \\ &= \underbrace{xx \cdots xx} + \underbrace{xx \cdots xy} + \underbrace{xx \cdots yx} + \underbrace{xx \cdots yy} + \cdots + \underbrace{yy \cdots yy} \end{aligned}$$

Pick one letter from each bracket to form a product of  $n$  letters.  
Sum up all product formed in this way.

There are  $C_k^n$  ways to choose  $k$  copies of  $x$  from among the  $n$  brackets. For each such choice we get a term of the form  $x^k y^{n-k}$

# Sums

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \quad \leftarrow \text{This is the } n\text{-th triangular number}$$

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + (n-1)n = \frac{(n-1)n(n+1)}{3}$$

Use the 2x Hockey Stick formula:  $2(C_2^2 + C_2^3 + C_2^4 + \dots + C_2^n) = 2C_3^{n+1}$

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + (n-2)(n-1)n = \frac{(n-2)(n-1)n(n+1)}{4}$$

Use the 3x Hockey Stick formula:  $3(C_3^3 + C_3^4 + \dots + C_3^n) = 3C_4^{n+1}$

# Sums

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Check this formula in three ways:

1. By Induction.

2. Use  $1^2 + 2^2 + \dots + n^2 =$

$$= 1 + 2(1 + 1) + 3(2 + 1) + 4(3 + 1) + \dots + n(n-1 + 1) =$$

$$= 1 + 2 + 3 + \dots + n + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + \dots + n(n-1)$$

Then use the sum formulas on the previous page.

3.  $1^3 = 1$

$$2^3 = (1 + 1)^3 = 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1$$

$$3^3 = (2 + 1)^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$$

$$4^3 = (3 + 1)^3 = 3^3 + 3 \cdot 3^2 + 3 \cdot 3 + 1$$

.....

$$(n+1)^3 = n^3 + 3 \cdot n^2 + 3 \cdot n + 1$$

Add these up by columns and cancel cubes.