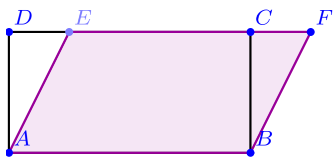
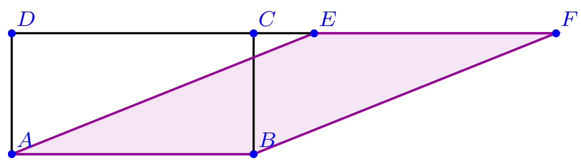
Areas

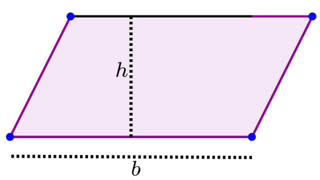
Quite a few ancient and famous theorems can be proven using only the area of a rectangle and congruent triangles.

(1) $Area(ABCD) = 10\,\mathrm{cm}^2$. Find each of the shaded areas. Here $AE \parallel BF$.

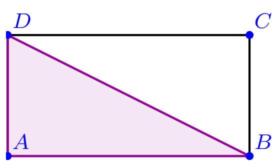


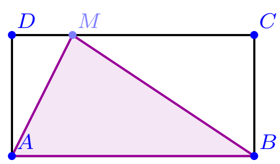


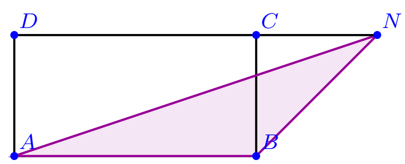
Thus we obtain the formula for $\boxed{\text{the area of a parallelogram } = h \cdot b}$.



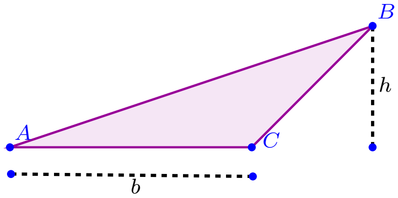
(2) $Area(ABCD) = 10\,\mathrm{cm}^2$. In each case find the shaded area.



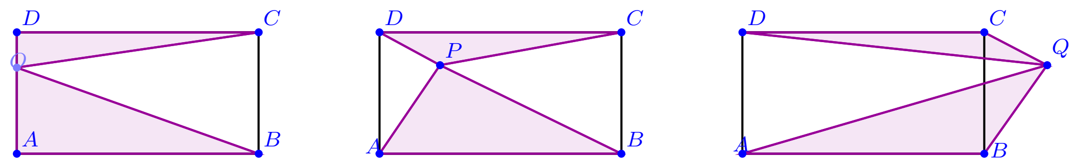




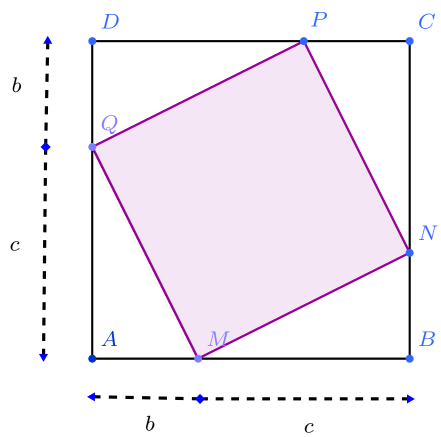
Thus we obtain the formula for $\boxed{\text{the area of a triangle } = \dfrac{h \cdot b}{2}}$.



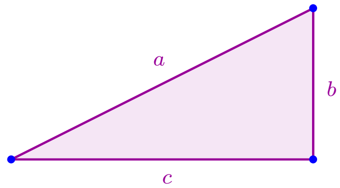
(3) $Area(ABCD) = 10\,\mathrm{cm}^2$. In each case find the total shaded area.



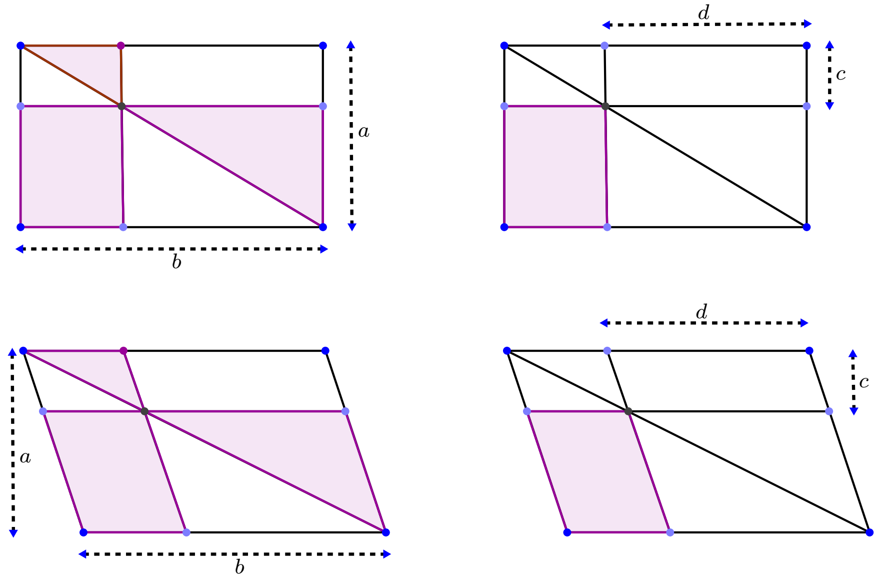
(4) Here $ABCD$ is a square. Assuming b and c to be some known numbers, find the shaded area. Simplify your formula as much as possible.



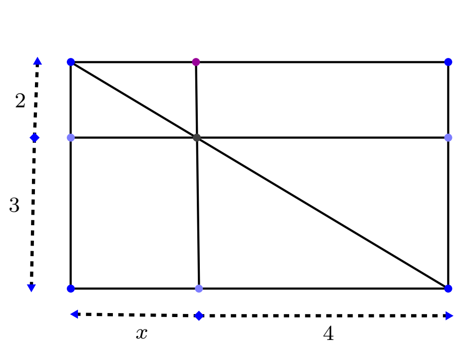
Thus we obtain Pythagora's Theorem: $\boxed{a^2 = b^2 + c^2}$.

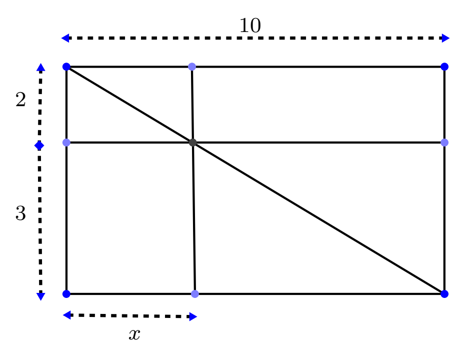


(5) Find the shaded areas in terms of a, b, c and d:

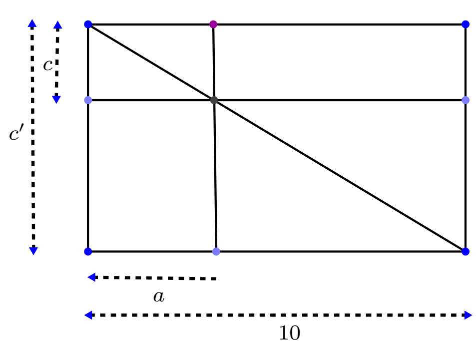


(6) Find x in each of the following figures:





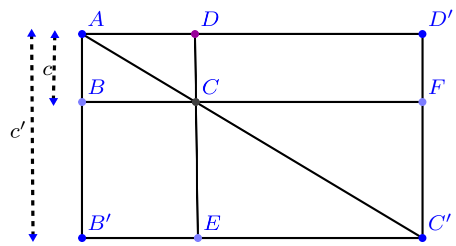
(7) Here we know $\frac{c}{c'} = \frac{1}{3}$. Find a.



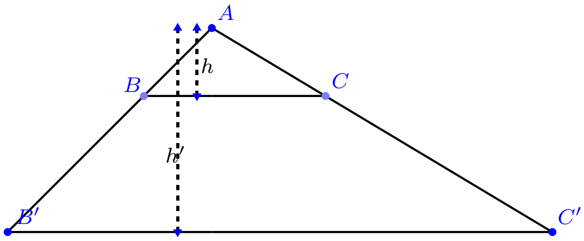
(8) Some useful algebra: knowing that $\boxed{\dfrac{c}{c'} = \dfrac{a}{a'}} = t$, prove that

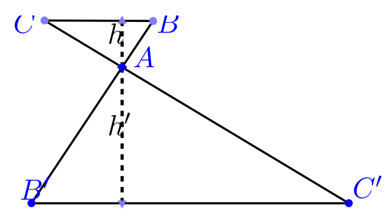
$$\frac{c}{c'} = \frac{a}{a'} = \frac{c-a}{c'-a'} = \frac{c+a}{c'+a'} = \frac{\sqrt{c^2+a^2}}{\sqrt{c'^2+a'^2}} = \sqrt{\frac{ac}{a'c'}} = t.$$

(9) Here we know $\frac{|AB|}{|AB'|} = \frac{c}{c'}$. Find $\frac{|BC|}{|B'C'|}$ and $\frac{|AC|}{|AC'|}$. You may denote $|BC| = a$ and $|B'C'| = a'$.



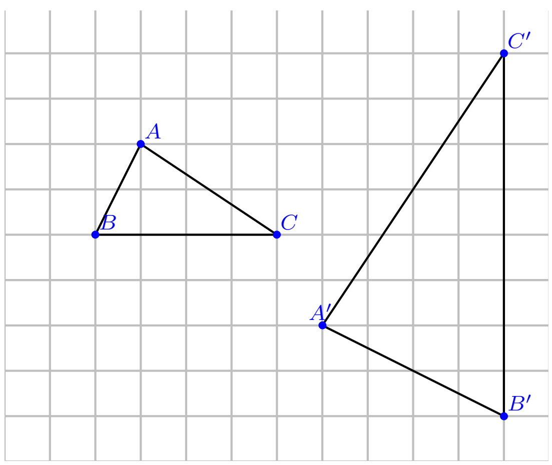
(10) Here $B'C' \parallel BC$. Write $\frac{|AB|}{|AB'|}$, as well as $\frac{|AC|}{|AC'|}$ and $\frac{|BC|}{|B'C'|}$ in terms of h and h'.





Definition: Similar triangles. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are called similar, and denoted $\boxed{\triangle ABC \sim \triangle A'B'C'}$, if their respective angles are equal:

$$\hat{A} = \hat{A}',\ \hat{B} = \hat{B}',\ \hat{C} = \hat{C}'.$$



Theorem: Similar triangles. Similar triangles have proportional sides:

$$\triangle ABC \sim \triangle A'B'C' \iff \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}.$$

Corollary: Trigonometric functions The trigonometric functions ($\sin A$, $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\csc A$) depend only on the measure of the angle $\hat{A}$, and are independent on the triangle in which A is embedded.

(11) ***Theorem: S.A.S for similar triangles.*** Two triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy:

$$\triangle ABC \sim \triangle A'B'C' \iff \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} \text{ and } \hat{A} = \hat{A}'.$$

(12) ***The Midline Theorem:***

If ABC is a triangle and B' is the midpoint of the segment AB, and C' is the midpoint of the segment AC, then $B'C' \parallel BC$ and $|B'C'| = \frac{1}{2}|BC|$.

$B'C'$ is called **midline** in $\triangle ABC$.

(13) Let ABC be any triangle, let M be a point on the side BC, and let L be a point on the line AM. Prove that

$$\frac{[ABL]}{[ACL]} = \frac{|BM|}{|CM|}.$$

A ***median*** is the line connecting a vertex of a triangle with the midpoint of the opposite line. This exercise proves in particular the ***Median Property:***

Let ABC be any triangle, let M be a point on the side BC, and let L be any point on the line AM. Then the triangles $\triangle ABL$ and $\triangle ACL$ have the same area if and only if AM is median.

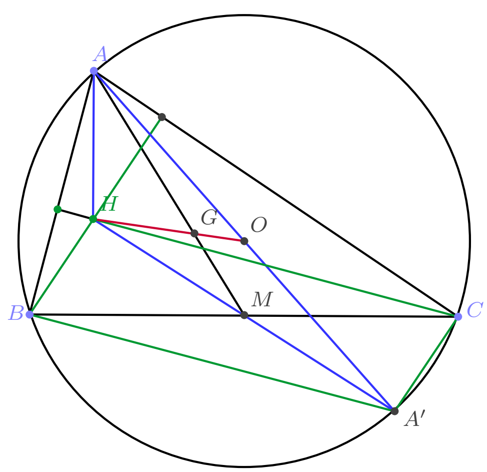
(14) ***Theorem: The Centroid.***

Let ABC be any triangle, let G be the intersection of two medians. Then G is also on the third median.

G is called the Centroid of triangle ABC.

(15) Let ABC be any triangle and G its centroid. Let M be the midpoint of the side BC. Check that $|AG| = 2|MG|$.

(16) ***Theorem: Euler's line.*** The orthocentre H, circumcentre O, and centroid G of any triangle $\triangle ABC$ are collinear and satisfy $|HG| = 2|HO|$.



Proof:

a) Let M be the midpoint of the segment BC and let AA' be the diameter of the circumcircle of $\triangle ABC$. Then $BHCA'$ is a parallelogram.

b) Hence M is the midpoint of segment $[HA']$.
c) Hence G is the centroid of $\triangle AHA'$. As such, G is on the median HO and $|HG| = 2|HO|$ by the Theorem of the Centroid.

Other area exercises:

(17) Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $|AE| = 7|EB|$. What is the ratio of the area of $EFGH$ to the area of $ABCD$?

(18) A square $ABCD$ of side length 10 cm, with E, F, G and H the midpoints of the sides BC, CD, DA and AB respectively, is broken into puzzle pieces by cutting out along a set of lines. For each of the following three puzzles, find the areas of the pieces and the lengths of their sides:
(a) The cutting lines are: AF, BG, CH, DE.
(b) The cutting lines are: AE, AC, BF.
(c) The cutting lines are: AE, AC, BF, BD.

(19) Let $n \geq 4$ be an integer. A point X in the interior of a square region R is called n–ray partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100–ray partitional but not 60–ray partitional?

(20) Let ABC be a triangle of area 10 cm^2, and let P denote the midpoint of the side BC. Consider two points M and N interior to the sides AB and AC respectively, such that $AM = 2MB$ and $CN = 2AN$. The lines AP and MN intersect at a point D. Find the area of the triangle ADN.

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