AM-GM-HM Inequalities

1. Prove the following:

a) $\frac{1}{2}+\frac{2}{3}+\cdots+\frac{999}{1{,}000}\geq\frac{999}{\sqrt[333]{10}}$

b) $\left(\frac{2}{1}\right)^2\left(\frac{4}{3}\right)^2\left(\frac{6}{5}\right)^2\left(\frac{8}{7}\right)^2\ldots\left(\frac{1{,}000}{999}\right)^2>\ 1{,}000.$

Hint: For all the even factors x, use $x^2>x^2-1=(x-1)(x+1)$.

c) $\frac{2}{1}+\frac{4}{3}+\frac{6}{5}+\frac{8}{7}+\cdots+\frac{1{,}000}{999}>\ 500\ \sqrt[1{,}000]{1{,}000}$

2. Let $x,y>0$. Prove the following:

a) $\frac{x}{y}+\frac{y}{x}\geq 2.$

b) $x^2+y^2\geq\frac{1}{2}(x+y)^2.$

c) $\frac{xy}{x+y}\leq\frac{x+y}{4}.$

3. Let $a,\ b,\ c\geq 0$. Prove the following:

a) $a+b+c\geq 3\sqrt[3]{abc}.$

b) $ab+bc+ca\geq\ 3\sqrt[3]{a^2b^2c^2}.$

c) $(a+b+c)(\ ab+bc+ca)\geq 9\ abc.$

d) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9.$

e) $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}.$

Hint: In (e), replace a by $b+c$, replace b by $a+c$, and c by $a+b$.

f) $8abc\leq(a+b)(a+c)(b+c)\leq\frac{8}{27}(a+b+c)^3.$

g) $abc(a+b+c)\leq a^4+b^4+c^4.$

4. Let $x_1,x_2,\ldots,x_n\geq 0$. Prove the following:

a) $(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_3}\right)\geq n^2.$

b) $\frac{x_1+x_2}{x_3+x_4}+\frac{x_2+x_3}{x_1+x_4}+\frac{x_3+x_4}{x_1+x_2}+\frac{x_4+x_1}{x_3+x_2}\geq 4.$

c) $\frac{x_1}{x_2+x_3+x_4}+\frac{x_2}{x_1+x_3+x_4}+\frac{x_3}{x_1+x_2+x_4}+\frac{x_4}{x_2+x_3+x_1}\geq\frac{4}{3}$

5. Let $a,\ b,\ c,\ d\geq 0$. Prove that

$$a^2(2b+c)+b^2(2c+a)+c^2(2a+b)-abc\geq(a+b)(a+c)(b+c).$$