UCC Mathematics Enrichment – Combinatorics

In how many ways can you select k objects from among n given objects in each case?

	Order of selection is important	Order is not important
Repetitions are not allowed		
Repetitions are allowed		

1. The letters U,V,W,X,Y,Z are used to form strings of length 4.

a) How many strings can be formed if we allow some repeated letters?
b) How many strings can be formed if we do not allow repetition?
c) How many strings contain at least one letter repeated at least once?
d) How many strings can be formed, not allowing repetition, if we must include the X?

2. Suppose 7 shoes are taken at random from 8 different pairs.

a) How many ways can this be done?
b) How many ways can this be done so that you have 3 pairs?
c) How many ways can this be done so that you have exactly 1 pair?
d) What is the probability that you have taken an odd number of pairs of shoes? What is the probability that you have taken an even number of pairs? e) Given that you have an odd number of pairs, what is the probability that it is 3 pairs?

3. In each case, find the number of ways of placing 6 marbles in 10 distinguishable boxes:
a) The marbles are distinguishable, and no box can hold more than one marble.
b) The marbles are indistinguishable, and no box can hold more than one marble.
c) The marbles are distinguishable, and each box can hold any number of them.
d) The marbles are indistinguishable, and each box can hold any number of them.

4. How many strings can be formed by ordering the letters $ABCDEFGH$ in each of the following ways:

i) containing the substring $ACEG$?
ii) containing the letters A, C, E and G together in any order?
iii) such that A appears before C and C appears before E and E appears before G?
iv containing neither of the substrings AC, EG?

5. How many strings can be formed in each case by ordering the given letters:

i) $SALESPERSONS$;
ii) $SALESPERSONS$ if no two S's are consecutive;
iii) $SALESPERSONS$ if all four S's are consecutive;
iv) $SALESPERSONS$ if exactly three S's are consecutive;
v) $SALESPERSONS$ if they contain two pairs of two consecutive S's;
vi) $SALESPERSONS$ if they contain exactly one pair of two consecutive S's?

6. Find the coefficient of X^5 in the polynomial $(1+X+X^2+X^3+X^4+X^5)^4$.
Hint: When you multiply the 4 brackets, you pick one power of X from each bracket. Use $\star$-s to remember the chosen power and $|$-s to separate the brackets.

7. For fixed positive numbers k and n, find:

a) the number of solutions in nonnegative integers to the equation $x_1 + x_2 + ... + x_k = n$.
b) the number of solutions in positive integers to the equation $x_1 + x_2 + ... + x_k = n$.
c) the number of solutions in nonnegative integers to the equation $x_1 + x_2 + ... + x_k \leq n$.
d) the number of sequences $0 \leq x_1 \leq x_2 \leq ... \leq x_k \leq n$.
e) the number of sequences $0 < x_1 < x_2 < ... < x_k < n$.
f) the number of sequences $0 < x_1 < x_2 < ... < x_k < n$ so that no two numbers in the sequence are consecutive.

Combinatorial identities

6. A composition of a number n with k parts is a sequence $\bar{x} = (x_1, x_2, ..., x_k)$ of positive integers such that $\sum_{i=1}^{k} x_i = n$. For $n \geq 2$, prove the following:

a) The number of all compositions of n is 2^{n-1}.
b) The total number of parts of all compositions of n is $(n+1)2^{n-2}$.
c) The number of compositions of n with an even number of parts is 2^{n-2}.

In the following problems, m, n, p are fixed positive whole numbers.

7. Let $n > k$ be two positive integers. Prove

$$\binom{n}{k} + \binom{n-1}{k} + ... + \binom{k}{k} = \binom{n+1}{k+1}.$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + ... + (-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k}.$$

8. Prove the following by two methods: $\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{m+n}{p}$.

Hence or otherwise, prove

a) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

b) $\sum_{k=0}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.

c) $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}$.

9. Prove that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + ... = \frac{1}{3}(2^n + 2\cos\frac{n\pi}{3}).$$

10. (IMO81) Take r such that $1 \leq r \leq n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

11. Let $n > p + q$. Prove

$$\binom{q}{q}\binom{n-q}{p} + \binom{q+1}{q}\binom{n-q-1}{p} + \ldots$$

$$\ldots + \binom{n-p-1}{q}\binom{p+1}{p} + \binom{n-p}{q}\binom{p}{p} = \binom{n+1}{p+q+1}.$$

The Inclusion–Exclusion principle

12. A DNA strand is represented by a sequence made out of 4 letters A, C, G, T. Find the number of DNA strands of length n which do not contain the string TA.

13. a) Find the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 10$ under the condition that $0 \leq x_i \leq 5$ for each i.
b) More generally for fixed k, a and b such that $a < b$ and $bk \geq n$, find the number of solutions in nonnegative integers to $x_1 + x_2 + ... + x_k = n$ under the condition that $a \leq x_i \leq b$ for each i.

14. For each positive number n, let $\varphi(n)$ denote the number of positive integers smaller than n which are relatively prime with n (i.e which have no common factors with n).

a) If $n = p^k$ for a prime number p, prove that $\varphi(n) = n(1 - \frac{1}{p})$.
b) If $p_1, p_s, ..., p_s$ are all the distinct prime factors of n, prove that

$$\varphi(n) = n \prod_{i=1}^{s} (1 - \frac{1}{p_i}).$$

15. Let $[n] = \{1, 2, ..., n\}$ and consider a k-tuple $A_1, A_2, ..., A_k$ of disjoint subsets of $[n]$ whose union is the entire set $[n]$.

a) Find the number k-tuples as above, if empty subsets are allowed.
b) Find the number k-tuples as above, if empty subsets are not allowed.
c) Let $S(n, k)$ denote the number of ways to partition a set of n elements into k disjoint, non-empty subsets (the order of the subsets is not important). Write a formula for $S(n, k)$.
d) Find the number of surjective functions from $[n]$ to a set with k elements.

16. n people enter a club carrying umbrellas. In how many ways can the umbrellas be returned to them at the end of the evening? In how many ways can the umbrellas be returned to them in such a way that no one gets their own umbrella back? (these last are called *derangements*).

17. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_{k=0}^{n} k p_n(k) = n!$

18. A permutation $\{x_1, x_2, \ldots, x_m\}$ of the set $\{1, 2, \ldots, 2n\}$ where n is a positive integer is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n - 1\}$. Show that for each n there are more permutations with property P than without.

19. Show that the sum $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$ is 0 if n is odd, and equals $(-1)^{n/2} \binom{n}{n/2}$ for n even.

20. Show that the sum $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3$ is 0 if n is odd, and equals

$$(-1)^{n/2} \frac{(3n/2)!}{((n/2)!)^3}$$

for n even.