EGMO and UCD Enrichment Programme in Mathematics

Selection Test 4 February 2017

1. Triangle ABC has area S. Denote by M, N and P the midpoints of BC, CA and AB respectively. Prove that

$$2S\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}\right) \leq AM+BN+CP < \frac{3}{2}(AB+BC+CA).$$

2. A positive integer is said to be *near-square* if it is a product of two positive integers differing by 1. For example, 20 is a near-square because $20 = 4 \times 5$. Prove that every near-square positive integer can be expressed as the ratio of two other near-square positive integers.

3. Alice and Bob play a game with a string of 2017 pearls. In the first move, Alice cuts the string and Bob chooses a part. Thereafter, the player who chose a part at the end of a move will cut the string in the next move. A player loses if he or she obtains a string with a single pearl such that no more cuts are possible.

   Which of the two players has a winning strategy?

4. The diagonals of the convex quadrilateral $ABCD$ of area 1 intersect at O. If $\frac{BO}{DO} = \frac{1}{2}$ and $\frac{AO}{CO} = \frac{3}{4}$, find the area of triangles AOB, BOC, COD and DOA.

5. Determine with proof all prime numbers p for which $7p+4$ is the square of an integer.

6. (a) Simplify $(x^2-1)^2+(x^2+2x)^2-(x^2+x+1)^2$ and then factor the result as far as possible.

   (b) Show that there are infinitely many pairs of integers m, n for which m^2+n^2-mn is the square of an integer.

7. ABC is an acute triangle. The angle bisector AL, the altitude BH and the median CM are such that $\angle CAL = \angle ABH = \angle ACM$. Find the angles of triangle ABC.

8. The function μ is defined on the set of positive integers as follows:

   - $\mu(1) = 1$ and $\mu(p) = -1$ for any prime number p;
   - $\mu(ab) = \mu(a)\mu(b)$ for any positive integers with $\gcd(a, b) = 1$;
   - $\mu(n) = 0$ if n is a positive integer which is divisible by a square of a prime number.

   (For instance $\mu(15) = \mu(3)\mu(5) = 1$ and $\mu(12) = 0$, because 12 is divisible by 2^2).

   Prove that for any positive integer $n > 1$, we have $\displaystyle\sum_{d|n} \mu(d) = 0$.

9. Let p, q, r be prime numbers with

$$p < q < r < q + p^4 \quad \text{and} \quad pq^2 = r^2 + 1.$$

   Find, with proof, all possible values for p, q and r.

10. The positive real numbers a, b, c satisfy the double inequality

$$\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a} \geq \frac{c^2}{a+b}+\frac{a^2}{b+c}+\frac{b^2}{c+a} \geq \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}.$$

   Prove that $a = b = c$.