In how many ways can we select *k* objects from among *n* given objects?

	Order of selection is important	Order is not important
Repetition is allowed		
Repetition is not allowed		

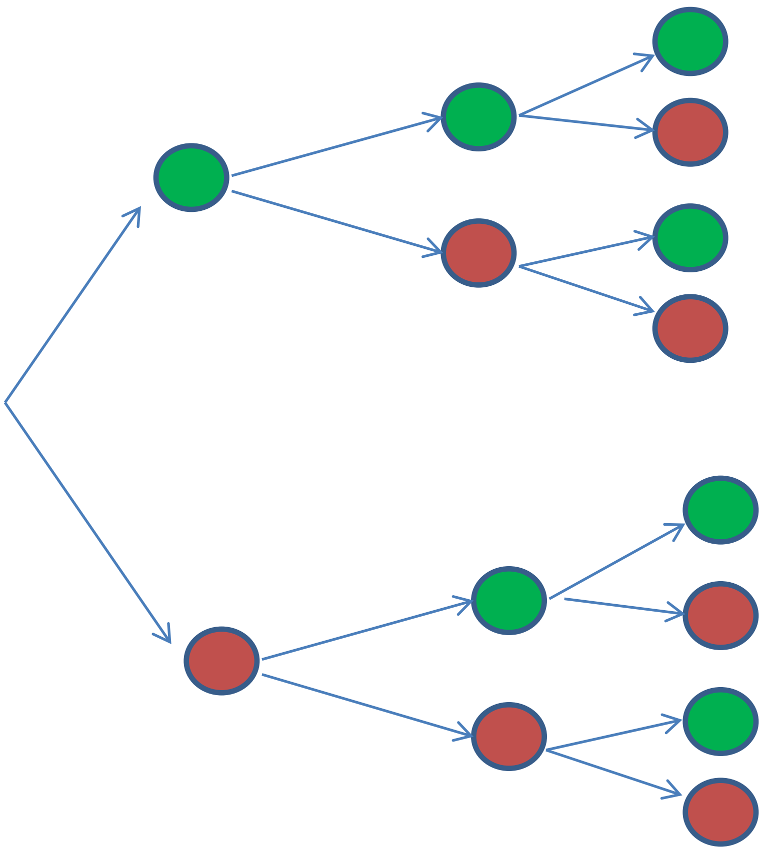
Today we discuss how to fill in these four slots.

How many colour codes can be made…

1. a) Codes of length 3 when 2 colours are available.

How many colour codes are there...

1. a) Codes of length 3 with 2 colours available.

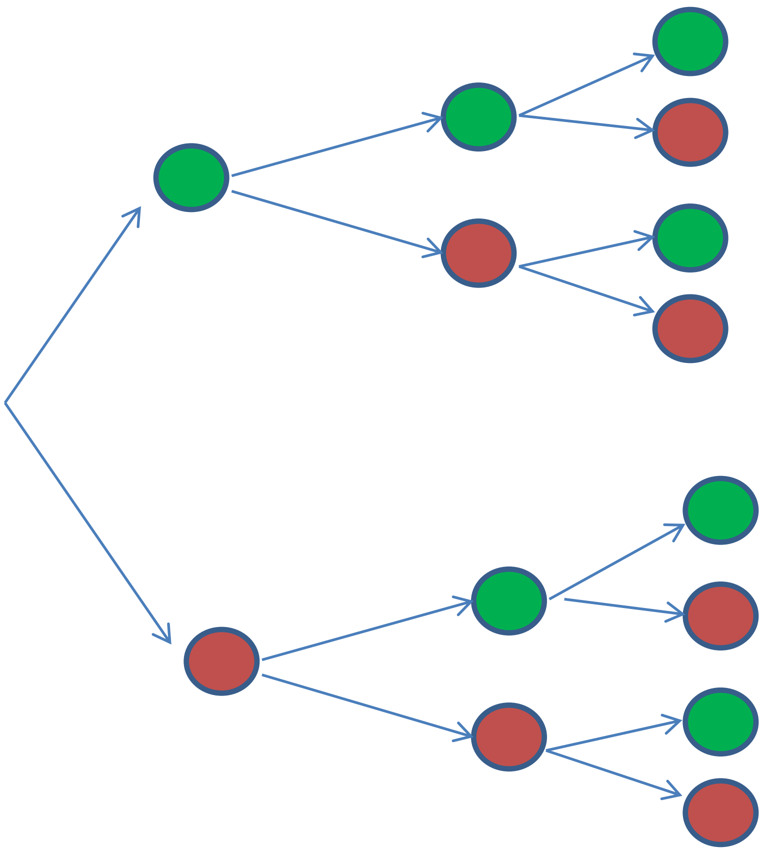


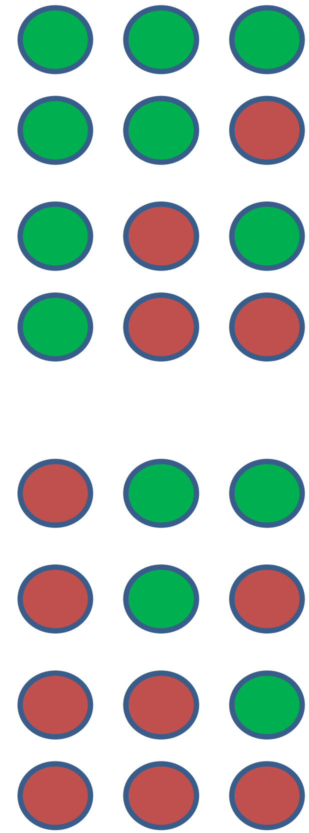
1^{st} colour 2^{nd} colour 3^{rd} colour

2 options $\times$ 2 options $\times$ 2 options

How many colour codes are there...

1. a) Codes of length 3 with 2 colours available.





1st colour	2nd colour	3rd colour	
2 options $\times$	2 options $\times$	2 options	$= \quad 2^3 = 8$

How many colour codes can be made ...

1.b) Codes of length *4* when these colours are available and repetitions are allowed? 

How many colour codes can be made ...

1.c) Codes of length k when n colours are available and repetitions allowed.

How many colour codes are there...

1.c) Codes of length *k* from *n* colours available when repetitions allowed.

$$\underbrace{n}_{\substack{\text{options}\\ \text{for}\\ 1^{\text{st}}\text{ colour}}} \times \underbrace{n}_{\substack{\text{options}\\ \text{for}\\ 2^{\text{nd}}\text{ colour}}} \times \ldots \times \underbrace{n}_{\substack{\text{options}\\ \text{for}\\ k^{\text{th}}\text{ colour}}} = n^k$$

How many colour codes can be made...

2. a) Codes of length 3 when 5 colours are available and no repetitions are allowed.

How many colour codes are there...

2. a) Codes of length 3 when 5 colours are available and no repetitions are allowed.

$$5 \times 4 \times 3 = 60$$

options for the 1^{st} colour

options for the 2^{nd} colour

options for the 3^{rd} colour

How many colour codes are there...

2. b) Codes of length k when n colours are available and no repetitions are allowed:

How many colour codes are there...

2. b) Codes of length k when n colours are available and no repetitions are allowed:

$$n(n-1)(n-2)\cdots(n-k+1) = P_k^n = \frac{n!}{(n-k)!}$$

options for the 1st colour

options for the 2nd colour

options for the 3rd colour

options for kth colour

k factors in all

How many colour codes are there...

2. c) Obtained by arranging all these colours: 
in all possible orders?

How many colour codes are there...

2. c) In particular there are $k! = k(k-1)(k-2)\cdots 1$

ways to arrange k colours in a row.

You may also think of it this way: you have k empty spaces for your k colours:

$$\_\_\ \_\_\ \_\_\ \_\_\ \cdots\ \_\_$$

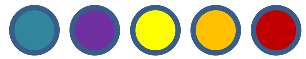
Choose a place for the 1st colour: k options.
Choose a place for the 2nd colour: $(k-1)$ options.
Choose a place for the 3rd colour: $(k-2)$ options.

……………………………………………………………………

Choose a place for the kth colour: 1 option.

How many **sets** of colours are there...
The order is not important

3. a) In how many ways can I choose a set of 3 colours from among these?

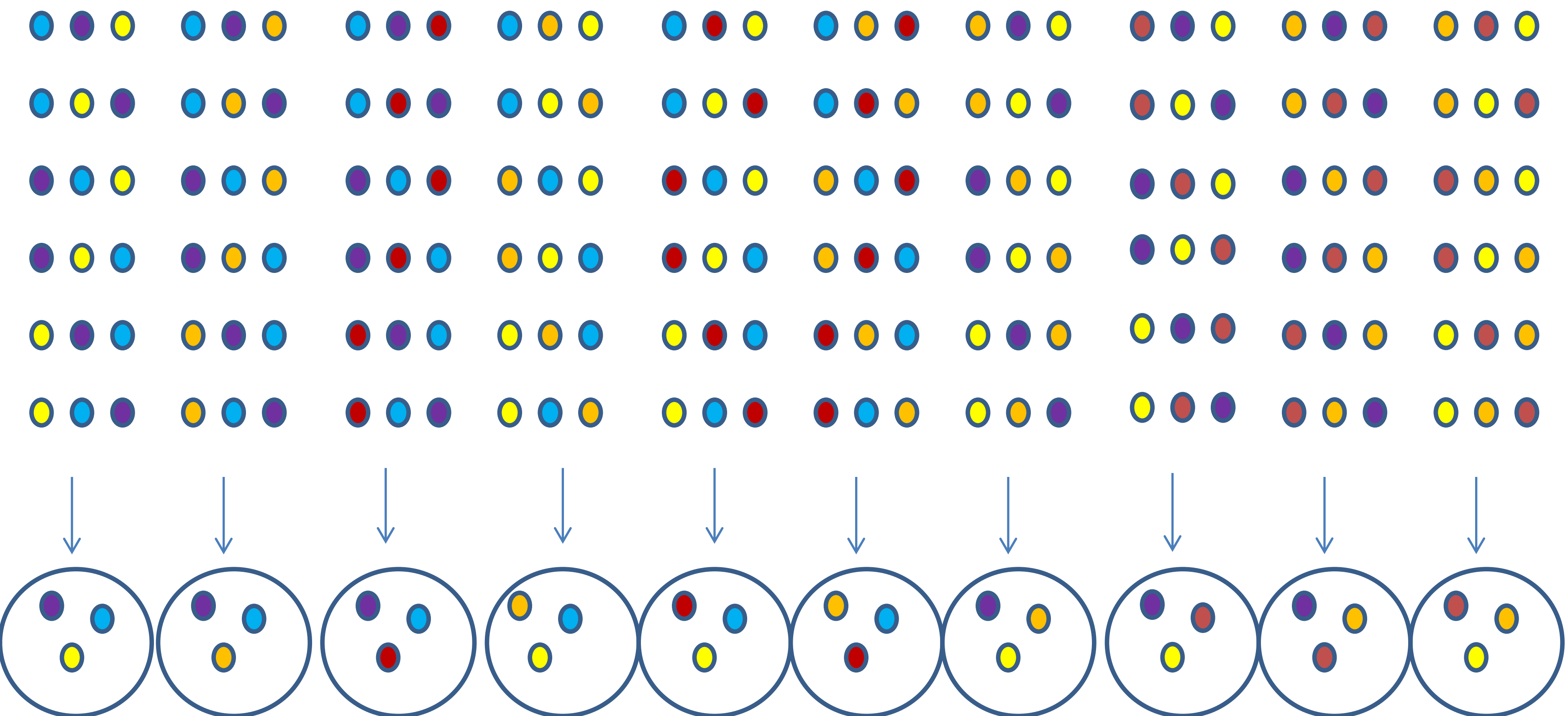


The order of selection is not important.

How many **sets** of colours are there…
The order is not important

3. a) In how many ways can I choose a set of 3 colours from among these?

The order of selection is not important.



How many **sets** of colours are there... The order is not important

3. b) How many sets of *k* objects selected from among *n* objects:

$$\binom{n}{k} = C_k^n = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}$$

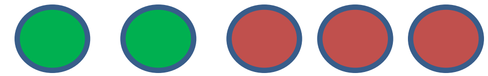
$$C_k^n = \frac{n!}{k!\,(n-k)!}$$

The top (numerator) represents all the ways in which you can choose your objects in order.

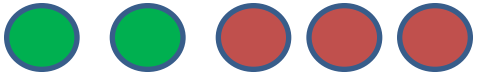
The bottom (denominator) represents all the ways in which the chosen objects can be ordered.

There are k factors at the numerator and k at the denominator

How many colour codes are there...

3. c) Codes of length 5 with exactly 3 Reds and 2 Greens? 

How many colour codes are there...

3. c) Codes of length 5 with exactly 3 Reds and 2 Greens? 

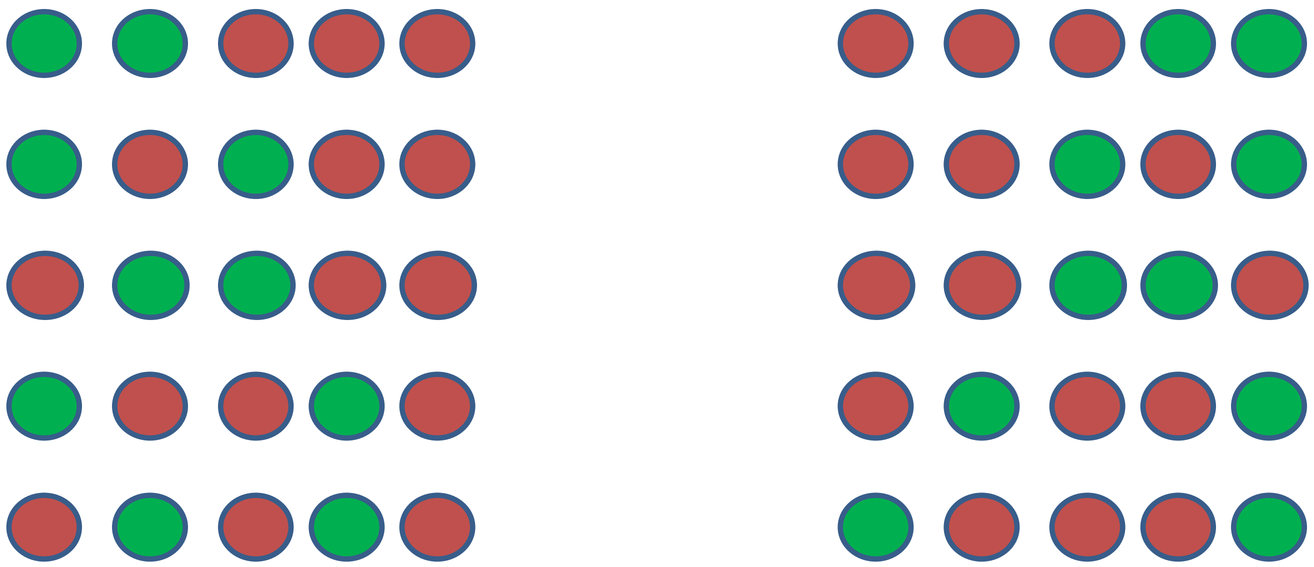
Imagine 5 empty spaces: ___ ___ ___ ___ ___

and choose 3 of them where to place your Reds. Then the Greens will go in the remaining places

How many colour codes are there...

Codes of length 5 with exactly 3 Reds and 2 Greens:

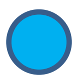
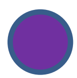
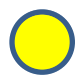
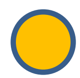
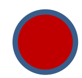
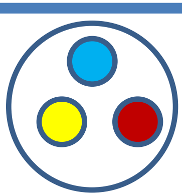
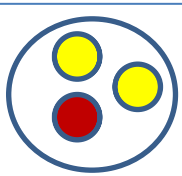
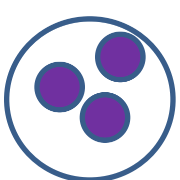
$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{60}{6} = 10$$



How many **multisets** are there?...

4. a) In how many ways can I choose a multiset of 3 colours from among these?

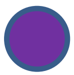
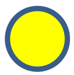
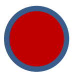
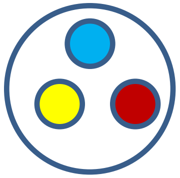
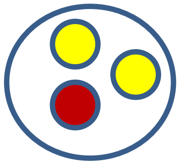
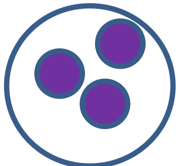
The order of selection is not important, and you may repeat the same element.

Examples:	Categories:					Notation:
						
						
						
						
						

How many **multisets** are there?...

4. a) In how many ways can I choose a multiset of 3 colours from among these?

The order of selection is not important, and you may repeat the same element.

<table>
<tr><th>Examples:</th><th colspan="5">Categories:</th><th>Notation:</th></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>★</td><td></td><td>★</td><td></td><td>★</td><td>★| |★| |★</td></tr>
<tr><td></td><td></td><td></td><td>★★</td><td></td><td>★</td><td>| |★★| |★</td></tr>
<tr><td></td><td>★</td><td>★</td><td>★</td><td></td><td></td><td>★|★|★| |</td></tr>
<tr><td></td><td></td><td>★★★</td><td></td><td></td><td></td><td>|★★★| | |</td></tr>
</table>

How many **multisets** are there?...

4. Can you recover the multisets from the notations on the right?

The order of selection is not important, and you may repeat the same element.

Examples:	Categories:					Notation:
						★||★||★
						||★★||★
						★|★|★||
						|★★★|||

How many **multisets** are there…

The order is not important, repetition is allowed

4. b) In how many ways can we select k objects from among n objects if order is not important, and we can repeat same object?

In how many ways can we distribute k objects into n categories?

How many **multisets** are there...

The order is not important, repetition is allowed

4. b) In how many ways can we select *k* objects from among *n* objects if order is not important, and we can repeat same object?

In how many ways can we distribute *k* objects into *n* categories?

$$\boxed{C_k^{k+n-1} = \binom{k+n-1}{k}}$$

We count codes of *k* ★ -s

n-1 | -s

In how many ways can we select k objects from among n given objects?

	Order of selection is important	Order is not important
Repetition is allowed	n^k	$C_k^{k+n-1} = C_{n-1}^{k+n-1}$
Repetition is not allowed	$P_k^n = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$	$C_k^n = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!\,(n-k)!}$

End of Lesson
Combinatorics 1

Additional Questions Below

How many colour codes are there...

4. a) Codes of length 3 with 5 colours available (including Red)
if repetitions are not allowed and Red must be used.

How many colour codes are there...

4. a) Codes of length 3 with 5 colours available (including Red) if repetitions are not allowed and Red must be used.

$$3 \times \underbrace{4 \times 3} = 36$$

Where will you place the Red?

What other colours will you place in the other two available places?

How many colour codes are there...

4. a) Codes of length 3 with 5 colours available (including Red) if repetitions are not allowed and Red must be used.

$$3 \times \underbrace{4 \times 3}_{} = 36$$

Where will you place the Red?

What other colours will you place in the other two available places?

$$\underbrace{5 \times 4 \times 3}_{} - \underbrace{4 \times 3 \times 2}_{} = 36$$

All length 3 codes of 5 colours available

All length 3 codes of 4 colours available (after excluding Red)

How many colour codes are there...

4. b) Codes of length k with n+1 colours available (including Red) if repetitions are not allowed and Red must be used.

How many colour codes are there...

4. b) Codes of length k with n colours available (including Red) if repetitions are not allowed and Red must be used.

$$P_k^n - P_k^{n-1} = k\, n(n-1)\cdots(n-k)$$

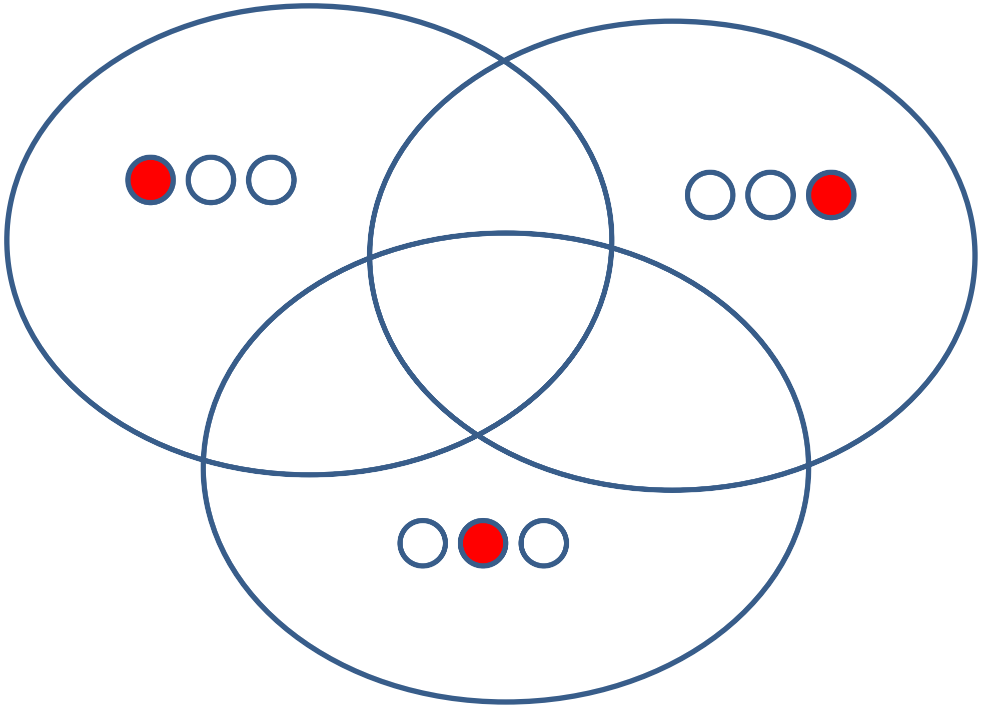
How many colour codes are there...

5. a) Codes of length 3 with 5 colours available (including Red) if repetitions are allowed and Red must be used.

Alternatively: Where could Red be in the code?

How many colour codes are there...

5. a) Codes of length 3 with 5 colours available (including Red) if repetitions are allowed and Red must be used.

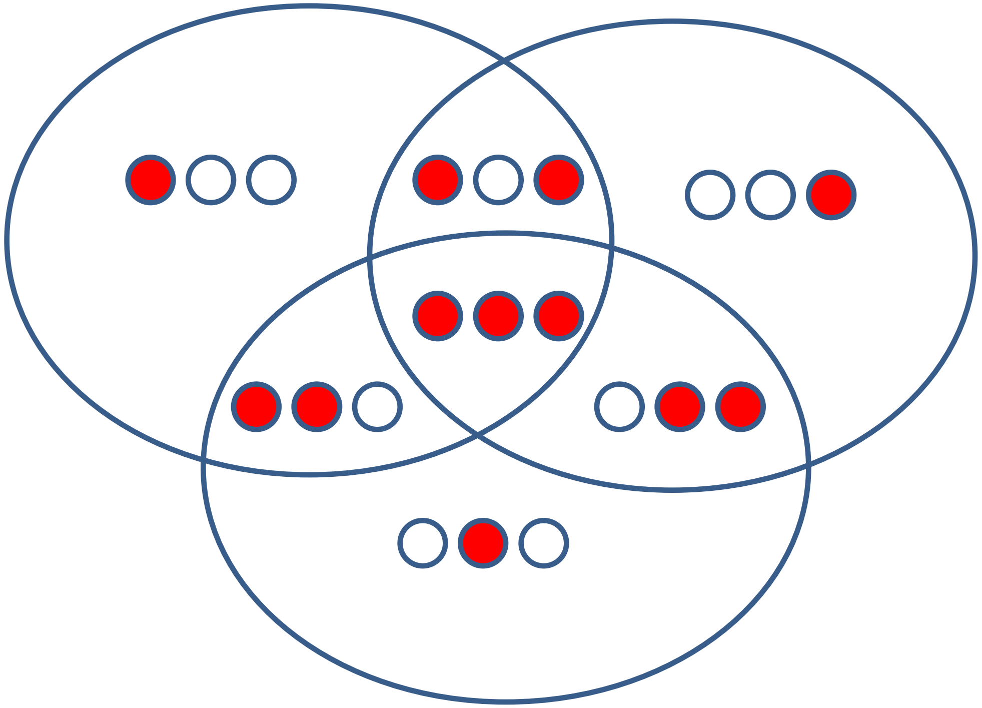


Where could Red be in the code?

How many colour codes are there...

5. a) Codes of length 3 with 5 colours available (including Red)

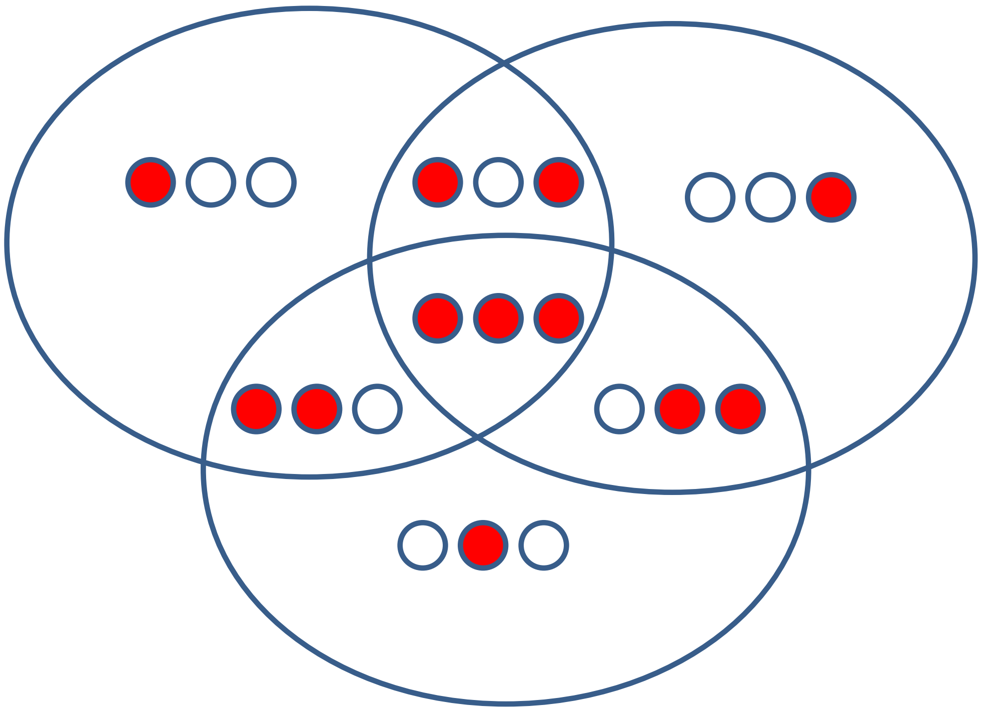
    if repetitions are allowed and Red must be used.



= any choice of colour except Red. How many elements in each set?

How many colour codes are there...

5. a) Codes of length 3 with 5 colours available (including Red)
   if repetitions are allowed and Red must be used.



 = any choice of colour except Red. How many elements in each set?

$$(4+1)^3 - 4^3 = 3 \times 4^2 + 3 \times 4 + 1$$

How many colour codes are there...

5. a) Codes of length k with $(n+1)$ colours available (including Red) if repetitions are allowed and Red must be used.

How many colour codes are there...

5. a) Codes of length k with (n+1) colours available (including Red) if repetitions are allowed and Red must be used.

All length *k* codes of *(n+1)* colours available

All length *k* codes of *n* colours available (after excluding Red)

How many colour codes are there...

5. a) Codes of length k with (n+1) colours available (including Red) if repetitions are allowed and Red must be used.

$$(n+1)^k - n^k$$

All length *k* codes of *(n+1)* colours available

All length *k* codes of *n* colours available (after excluding Red)

How many colour codes are there...

6\. a) Codes of length *5* with *7* colours available ( )

- repetitions are allowed and
- the codes must contain exactly *3* dark colours (are available)

  2 light colours (are available)

Before you fill in the available places with colours: __ __ __ __ __
you may want to choose which are the places for the dark/light colours.

How many colour codes are there...

6. a) Codes of length *5* with *7* colours available ( )

- repetitions are allowed and
- the codes must contain exactly *3* dark colours (are available)

2 light colours (are available)

Before you fill in the available places with colours: __ __ __ __ __
you may want to choose which are the places for the dark/light colours.

Choose *3* places out of the *5* available. That's where your dark colours will go.

For each of the *3* chosen places, choose one of the dark colours.

For each of the *2* remaining places, choose one of the *3* light colours.

How many colour codes are there…

6. a) Codes of length *5* with *7* colours available ( )

- repetitions are allowed and
- the codes must contain exactly *3* dark colours (are available)

  2 light colours (are available)

Before you fill in the available places with colours: __ __ __ __ __
you may want to choose which are the places for the dark/light colours.

$$C_3^5 \quad 4^3 \quad 3^2$$

Choose *3* places out of the *5* available. That's where your dark colours will go.

For each of the *3* chosen places, choose one of the dark colours.

For each of the *2* remaining places, choose one of the *3* light colours.

How many colour codes are there...

6. b) Codes of length n with $x+y$ colours (x dark colours and y light colours)
   - repetitions are allowed and
   - the codes must contain exactly k dark colours and $(n-k)$ light colours.

How many colour codes are there...

6. b) Codes of length n with $x+y$ colours (x dark colours and y light colours)
   - repetitions are allowed and
   - the codes must contain exactly k dark colours and $(n-k)$ light colours.

You must fill in n available places with $x+y$ colours. You must:

Choose k places out of the n available. That's where your dark colours will go.

For each of the k chosen places, choose one of the x dark colours.

For each of the n-k remaining places, choose one of the y light colours.

How many colour codes are there...

6. b) Codes of length *n* with *x+y* colours (*x* dark colours and *y* light colours)

- repetitions are allowed and
- the codes must contain exactly *k* dark colours and *(n-k)* light colours.

You must fill in *n* available places with *x+y* colours. You must:

$$C_k^n \quad x^k \quad y^{n-k}$$

Choose *k* places out of the *n* available. That's where your dark colours will go.

For each of the *k* chosen places, choose one of the *x* dark colours.

For each of the *n-k* remaining places, choose one of the *y* light colours.

How many colour codes are there...

6. c) Codes of length *n* with *x+y* colours (*x* dark colours and *y* light colours) if repetitions are allowed:

$$(x+y)^n = \sum_{k=0}^{n} C_k^n \, x^k y^{n-k}$$

$$= C_0^n x^0 y^n + C_1^n x^1 y^{n-1} + C_2^n x^2 y^{n-2} + \cdots$$

Codes with no light colours

Codes with 1 light colour

Codes with 2 light colours

How many colour codes are there...

Codes of length n=0, 1, 2, 3, 4, or 5, with x light colours and y dark colours available, repetitions allowed.

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2x^1y^1 + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y^1 + 3x^1y^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5$$